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# Lax Representations and What to do With Them

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# Outline of Lectures and Lecture Notes

Plan of the Lectures:

- (1) Lax Representations and What to Do with Them: an introduction to the inverse scattering method
- (2) Large-Time Asymptotics for the Defocussing NLS Equation: an introduction to asymptotic analysis by nonlinear steepest descent
- (3) Inverse Scattering in Two Dimensions: the Davey-Stewartson II Equation
- (4) Inverse Scattering in Two Dimensions: Scattering Theory and Asymptotics

An excellent reference for dispersive equations via inverse scattering and PDE methods is the book *Nonlinear dispersive equations–inverse scattering and PDE methods* by Christian Klein and Jean-Claude Saut

PDF lecture notes may be found at:

www.math.uky.edu/~perry/lectures.htm

# Acknowledgments

Parts of this lecture are taken from 2018 lecture notes of Peter Miller for a graduate course at the University of Michigan (see especially Topic 3 and Topic 5). Some key references are:

- 1. Vladimir Zakharov, A. B. Shabat. Exact theory of two-dimensional self-focussing and one-dimensional self-modulation of waves in nonlinear media. *Soviet Physics JETP* **34** (1972), no. 1, 62–29. Translated from Ž. Èksper. Teoret. Fiz. **71** (1976), no. 1, 118-134.
- 2. Mark Ablowitz, David Kaup, Alan Newell, Harvey Segur. The inverse scattering transform: Fourier analysis for nonlinear problems. *Studies in Applied Math* **53** (1974), 249-315
- 3. Percy Deift, Xin Zhou. Perturbation theory for integrable systems on the line. A case study. *Acta Mathematica* **188** (2002), 163-262.
- 4. Percy Deift, Xin Zhou. Long-time asymptotics for solutions of the NLS equation with initial data in a weighted Sobolev space. *Comm. Pure Appl. Math.* **56** (2003), no. 8, 1029-1077.

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#### The KdV Equation

The Korteweg-de Vries (KdV) equation

$$q_t + q_{xxx} - 6qq_x = 0$$

is one of the earliest examples of a completely integrable dispersive PDE. It admits a *Lax Representation*: given  $q = q(x, t) \in C^{\infty}(\mathbb{R} \times \mathbb{R})$ , there are differential operators L(t) and A(t) given by

$$L(t) = -\partial_x^2 + q$$
$$A(t) = -4\partial_x^3 + 6q\partial_x + 3q_x$$

so that the Lax equation

$$\frac{\partial}{\partial t}L(t) + [L(t), A(t)] = 0$$

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is equivalent the the KdV equation.



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#### Another Point of View

Equivalently, the Lax condition

$$\frac{\partial}{\partial t}L(t) = [A(t), L(t)]$$

is the compatibility condition for the system of equations

$$L(t)\psi = \lambda\psi \tag{1}$$

$$\psi_t = A(t)\psi \tag{2}$$

for  $\psi = \psi(x, t)$  to have a solution: From (1),

$$\frac{\partial}{\partial t}(L(t)\psi) = \dot{L}\psi + L\psi_t = \dot{L}\psi + LA\psi$$

while

$$\frac{\partial}{\partial t}(\lambda\psi) = \lambda\psi_t = \lambda A\psi = A(\lambda\psi) = AL\psi$$

which gives

$$\dot{L}\psi + (LA - AL)\psi = 0$$

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#### Zero-Curvature Condition

Let

 $w = \begin{pmatrix} \psi \\ \psi_x \end{pmatrix}$ 

Using  $\psi_{xx} = (q - \lambda)\psi$  to express higher-order derivatives of  $\psi$  in terms of  $\psi$  and  $\psi_x$  we obtain

$$\frac{\partial}{\partial x}w = \begin{pmatrix} 0 & 1\\ q - \lambda & 0 \end{pmatrix} w$$
$$\frac{\partial}{\partial t}w = \begin{pmatrix} -q_x & (2q + 4\lambda)\\ -q_{xx} + 2q^2 + 2\lambda q - 4\lambda^2 & q_x \end{pmatrix} w$$

The new form of the Lax equations is

$$\frac{\partial w}{\partial x} = \mathbf{U}w,$$
$$\frac{\partial w}{\partial t} = \mathbf{V}w$$



#### Zero-Curvature Condition

The compatibility condition for the system

$$\frac{\partial w}{\partial x} = \mathbf{U}w, \quad \frac{\partial w}{\partial t} = \mathbf{V}w$$

is obtained by cross-differentiation:

$$w_{xt} = \frac{\partial \mathbf{U}}{\partial t}w + \mathbf{U}\mathbf{V}w, \qquad w_{tx} = \frac{\partial \mathbf{V}}{\partial x}w + \mathbf{V}\mathbf{U}u$$

so we have the zero-curvature condition

$$\frac{\partial \mathbf{U}}{\partial t} - \frac{\partial \mathbf{V}}{\partial x} = [\mathbf{U}, \mathbf{V}]$$

In the case of KdV, this condition becomes

$$\begin{pmatrix} 0 & 0 \\ q_t + q_{xxx} - 6qq_x & 0 \end{pmatrix} = \mathbf{0}$$

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which is exactly the KdV equation



# KdV and Inverse Scattering

For the KdV equation:

- The first Lax equation (Schrödinger equation) defines a *reflection coefficient r, eigenvalues* {λ<sub>i</sub>}<sup>N</sup><sub>i=1</sub>, and *norming constants* {c<sub>i</sub>}<sup>N</sup><sub>i=1</sub>
- (2) The second Lax equation (time evolution) prescribes the linear evolution of *r* and {*c<sub>i</sub>*}, and the invariance of {λ<sub>i</sub>}

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(3) The *inverse scattering map* reconstructs the solution *q*(*x*, *t*) from the time-evolved scattering data

Eigenvalues of  $q|_{t=0}$  correspond to solitons.

# Dynamics of KdV via Inverse Scattering

KdV

Recall that the first Lax equation (Schrödinger equation) defines a *reflection* coefficient *r*, eigenvalues  $\{\lambda_i\}_{i=1}^N$ , and norming constants  $\{c_i\}_{i=1}^N$ 

From the spectral data we recover a classification of possible solutions:

- If *r* = 0, the scattering data consist of eigenvalues and norming constants, and the KdV solution is a pure soliton solution
- If  $r \neq 0$  and there are no eigenvalues, the KdV solution consists of pure radiation
- If  $r \neq 0$  and there are eigenvalues, the KdV solution consists of solitons and radiation

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### The AKNS-ZS System

We can begin with a linear equation  $\frac{\partial w}{\partial x} = \mathbf{U}w$  and attempt to find matrices **V** which lead to integrable systems.

The AKNS system is given by

ZS-AKNS

$$rac{\partial w}{\partial x} = \mathbf{U}w, \quad \mathbf{U} = -i\lambda\sigma_3 + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$$

where q and r are functions of x and t, and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This system, and integrable equations associated to it, were studied in a landmark paper by Ablowitz, Kaup, Newell and Segur (1974).

The case where  $r = \pm \overline{q}$  is called the Zakharov-Shabat system



# Cubic Nonlinear Schrödinger Equation

The Lax representation

$$\begin{aligned} \frac{\partial w}{\partial x} &= -i\lambda\sigma_3 w + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} w \\ \frac{\partial w}{\partial t} &= -i\lambda^2\sigma_3 w + \lambda \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} w + \begin{pmatrix} -\frac{1}{2}iqr & \frac{1}{2}iq_x \\ -\frac{1}{2}ir_x & \frac{1}{2}iqr \end{pmatrix} w \end{aligned}$$

gives rise to the coupled system

$$iq_t + \frac{1}{2}q_{xx} - q^2r = 0$$
$$-ir_t + \frac{1}{2}r_{xx} - r^2q = 0$$

Taking  $r = \pm \overline{q}$  we get the cubic nonlinear Schrödinger equation

$$iq_t + \frac{1}{2}q_{xx} \mp |q|^2 q = 0$$

which is *defocussing* for the - sign and *focussing* for the + sign.

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A Lax representation

Inverse Scattering

$$\frac{\partial w}{\partial x} = \mathbf{U}(q)w$$
$$\frac{\partial w}{\partial t} = \mathbf{V}(q)w$$

defines

- (1) A spectral problem which maps a given potential q to scattering data r
- (2) A *time evolution* of scattering solutions which determines how the scattering data evolve in time
- (3) A *Riemann-Hilbert problem* which defines a map from scattering data *r* to the potential *q*

This leads to a strategy for solving the associated nonlinear equation by *inverse scattering* 



#### NLS Equation: Direct Scattering Map

Spectral Problem:

$$rac{\partial \Psi}{\partial x} = -iz\sigma_{3}\Psi + egin{pmatrix} 0 & q \ \overline{q} & 0 \end{pmatrix} \Psi$$

Fix *z*. One can show that:

- (1) det  $\Psi(x)$  is constant for any solution
- (2) If  $\psi_1$  and  $\psi_2$  are solutions,  $\psi_1(x) = \psi_2(x)M$  for a constant matrix M

(3) The map

$$\psi(x,z) \mapsto \sigma_1 \overline{\psi(x,\overline{z})} \sigma_1^{-1}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

preserves the solution space

For  $q = 0, z = \lambda \in \mathbb{R}$ ,  $\Psi(x, \lambda) = e^{-i\lambda\sigma_3 x}\Psi(0)$  are exact solutions.

For  $q \neq 0$ ,  $z = \lambda \in \mathbb{R}$ , look for solutions  $\Psi^{\pm}$  satisfying

$$\lim_{x \to \pm \infty} \Psi^{\pm}(x) e^{i\lambda x \sigma_3} = \mathbb{I}$$

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#### NLS Equation: Direct Scattering Map

Spectral Problem:

$$\frac{\partial \Psi}{\partial x} = -iz\sigma_3\Psi + \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix}\Psi$$

For  $z = \lambda \in \mathbb{R}$ , there exist unique solutions  $\Psi^{\pm}$  satisfying  $\lim_{x \to \pm \infty} \Psi^{\pm}(x) e^{i\lambda x \sigma_3} = \mathbb{I}$ .

From properties (2) and (3), there is a matrix

$$T(\lambda) = \begin{pmatrix} a(\lambda) & \overline{b(\lambda)} \\ b(\lambda) & \overline{a(\lambda)} \end{pmatrix}, \qquad |a(\lambda)|^2 - |b(\lambda)|^2 = 1$$

so that  $\Psi^+(x,\lambda) = \Psi^-(x,\lambda)T(\lambda)$ .

The functions  $a(\lambda)$ ,  $b(\lambda)$  are *scattering data* for q, and are uniquely determined by

$$r(\lambda) = -b(\lambda)/\overline{a(\lambda)}$$

The map  $\mathcal{R}$  :  $q \mapsto r$  is called the *direct scattering map* 

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### NLS: Direct Scattering Map

Deift and Zhou (2003) proved the following mapping property of the direct scattering map. Let

$$H^{1,1}(\mathbb{R}) = \{ u \in L^2(\mathbb{R}) : u', xu \in L^2(\mathbb{R}) \}$$
$$H^{1,1}_1(\mathbb{R}) = \{ u \in H^{1,1}(\mathbb{R}) : ||u||_{L^{\infty}} < 1 \}$$

#### Theorem (Deift-Zhou)

*The direct scattering map*  $\mathcal{R} : q \to r$  *is a Lipschitz continuous map from*  $H^{1,1}(\mathbb{R})$  *onto*  $H^{1,1}_1(\mathbb{R})$ .

A consequence is that, for  $q \in H^{1,1}(\mathbb{R})$ , the corresponding reflection coefficient satisfies the bound

$$|r(\lambda)| \le \rho$$

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for some  $\rho \in (0, 1)$ .

# NLS: Time Evolution of Scattering Data

Time-Evolution

Suppose now q = q(x, t) and that  $\Psi^{\pm}(x, t, \lambda)$  solves

$$\frac{\partial \Psi}{\partial x} = \mathbf{U}\Psi, \quad \mathbf{U} = -iz\sigma_3 + \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix}$$

We seek solutions

$$W^{\pm}(x,t,\lambda) = \Psi^{\pm}(x,t,\lambda)\mathbf{C}(t,\lambda)$$

of

$$\frac{\partial W^{\pm}}{\partial t} = \mathbf{V} W^{\pm}$$

By substitution

$$\frac{\partial \Psi^{\pm}}{\partial t} \mathbf{C}^{\pm} + \Psi^{\pm} \frac{\partial \mathbf{C}^{\pm}}{\partial t} = \mathbf{V} \Psi^{\pm} \mathbf{C}^{\pm}$$

or

$$\frac{\partial \mathbf{C}^{\pm}}{\partial t} = (\Psi^{\pm})^{-1} \mathbf{V} \Psi^{\pm} \mathbf{C}^{\pm} - (\Psi^{\pm})^{-1} \frac{\partial \Psi^{\pm}}{\partial t} \mathbf{C}^{\pm}$$

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Scattering Maps

# NLS: Time Evolution of Scattering Data

$$\frac{\partial \mathbf{C}^{\pm}}{\partial t} = (\Psi^{\pm})^{-1} \mathbf{V} \Psi^{\pm} \mathbf{C}^{\pm} - (\Psi^{\pm})^{-1} \frac{\partial \Psi^{\pm}}{\partial t} \mathbf{C}^{\pm}$$

Recall

$$\mathbf{V} = -i\lambda^2 \sigma_3 + \lambda \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}i|q|^2 & \frac{1}{2}iq_x \\ -\frac{1}{2}i\overline{q}_x & \frac{1}{2}i|q|^2 \end{pmatrix}$$

Assume

$$\Psi^{\pm}(x,t,\lambda) = e^{-i\lambda x\sigma_3} + \mathbf{E}^{\pm}(x,\lambda,t)$$

where

$$\mathbf{E}^{\pm}(x,\lambda,t), \ \frac{\partial \mathbf{E}^{\pm}}{\partial t}(x,\lambda,t) \to 0 \text{ as } x \to \pm \infty$$

Assume  $q, q_x \to 0$  as  $x \to \pm \infty$ . Taking  $x \to \pm \infty$  we conclude

$$\frac{\partial \mathbf{C}^{\pm}}{\partial t} = -i\lambda^2 \sigma_3 \mathbf{C}^{\pm}$$

so

$$\mathbf{C}^{\pm}(t,\lambda) = e^{-i\lambda^2 t\sigma_3}$$

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NLS: Time Evolution of Scattering Data

Time-Evolution

We will compute the time evolution of

$$T(\lambda,t) = (\Psi^{-}(x,t,\lambda))^{-1}\Psi^{+}(x,t,\lambda)$$

Since  $W^{\pm} = \Psi^{\pm} e^{-i\lambda^2 t\sigma_3}$  satisfies  $\frac{\partial W^{\pm}}{\partial t} = \mathbf{V}W^{\pm}$ , it follows that

$$\frac{\partial \Psi^{\pm}}{\partial t} = i\lambda^2 \Psi^{\pm} \sigma_3 + \mathbf{V} \Psi^{\pm}$$

Hence

$$\begin{aligned} \frac{\partial T(\lambda,t)}{\partial t} &= -(\Psi^{-})^{-1} \frac{\partial \Psi^{-}}{\partial t} (\Psi^{-})^{-1} \Psi^{+} + (\Psi^{-})^{-1} \frac{\partial \Psi^{+}}{\partial t} \\ &= -i\lambda^{2} \sigma_{3} T(\lambda,t) - (\Psi^{-})^{-1} \mathbf{V} \Psi^{+} + i\lambda^{2} T(\lambda,t) \sigma_{3} + (\Psi^{-})^{-1} \mathbf{V} \Psi^{+} \\ &= i\lambda^{2} [T(\lambda,t),\sigma_{3}] \end{aligned}$$

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# NLS: Time Evolution of Scattering Data

$$\frac{\partial T(\lambda, t)}{\partial t} = i\lambda^2 [T(\lambda, t), \sigma_3]$$
$$\frac{\partial}{\partial t} \begin{pmatrix} a & \overline{b} \\ b & \overline{a} \end{pmatrix} = i\lambda^2 \begin{pmatrix} 0 & -2\overline{b} \\ 2b & 0 \end{pmatrix}$$

or

Hence

$$a(\lambda, t) = a(\lambda, 0)$$
$$b(\lambda, t) = e^{2i\lambda^2 t}b(\lambda, 0)$$

so that

$$r(\lambda,t) = -\frac{b(\lambda,t)}{a(\lambda,t)} = e^{2i\lambda^2 t} r(\lambda,0).$$

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### NLS: Inverse Scattering Map

Recall the spectral problem

$$rac{\partial \Psi}{\partial x} = -iz\sigma_3 \Psi + egin{pmatrix} 0 & q \ \overline{q} & 0 \end{pmatrix} \Psi$$

and solutions  $\Psi^{\pm}$  with  $\lim_{x \to \pm \infty} e^{i\lambda x \sigma_3} \Psi^{\pm} = \mathbb{I}$ . For a solution  $\Psi$ , let

$$\Psi(x,z) = \mathbf{M}(x,z)e^{-izx\sigma_3}$$

so that

$$\frac{\partial}{\partial x}\mathbf{M}(x,z) = -iz \operatorname{ad}(\sigma_3)\mathbf{M}(x,z) + \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix} \mathbf{M}(x,z)$$

For a matrix A,

$$\mathrm{ad}(\sigma_3)A = [\sigma_3, A]$$

We denote by  $\mathbf{M}^{\pm}$  the normalized Jost solutions, i.e.,

$$\Psi^{\pm}(x,z) = \mathbf{M}^{\pm}(x,z)e^{-izx\sigma_3}$$

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#### NLS: Beals-Coifman Solutions

$$\frac{\partial}{\partial x}\mathbf{M}(x,z) = -iz \operatorname{ad}(\sigma_3)\mathbf{M}(x,z) + \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix} \mathbf{M}(x,z)$$

Beals and Coifman (1984) showed that this equation admits special solutions, now called the *Beals-Coifman solutions*, with the following properties:

- (i) For each  $x \in \mathbb{R}$ ,  $\mathbf{M}(x, z)$  is analytic in  $\mathbb{C} \setminus \mathbb{R}$
- (ii) For each x,  $\mathbf{M}(x, z)$  has continuous boundary values

$$\mathbf{M}_{\pm}(x,\lambda) = \lim_{\varepsilon \downarrow 0} \mathbf{M}(x,\lambda \pm i\varepsilon)$$

(iii) For each  $z \in \mathbb{C} \setminus \mathbb{R}$ ,

 $\mathbf{M}(x, z) \to \mathbb{I}$  as  $x \to +\infty$ ,  $\mathbf{M}(x, z)$  is bounded as  $x \to -\infty$ 

(iv) The potential q(x) can be recovered from their asymptotic behavior:

$$q(x) = \lim_{z \to \infty} 2iz \,\mathbf{M}_{12}(x, z)$$



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# NLS: Riemann-Hilbert Problem

Most importantly, the Beals-Coifman solutions satisfy a Riemann-Hilbert Problem:

**Riemann-Hilbert Problem 1**. For given *r* and  $x \in \mathbb{R}$ , find  $\mathbf{M}(x, z)$  so that:

- (i)  $\mathbf{M}(x, z)$  is analytic in  $\mathbb{C} \setminus \mathbb{R}$  for each  $x \in \mathbb{R}$
- (ii)  $\lim_{z\to\infty} \mathbf{M}(x,z) = \mathbb{I}$
- (iii)  $\mathbf{M}(x, z)$  has continuous boundary values  $\mathbf{M}_{\pm}(x, \lambda)$  on  $\mathbb{R}$
- (iv) The jump relation

$$\mathbf{M}_{+}(x,\lambda) = \mathbf{M}_{-}(x,\lambda)\mathbf{V}(x,\lambda), \quad \mathbf{V}(x,\lambda) = \begin{pmatrix} 1 - |r(\lambda)|^{2} & -\overline{r(\lambda)}e^{-2i\lambda x} \\ r(\lambda)e^{2i\lambda x} & 1 \end{pmatrix}$$

holds, where *r* is the scattering data.

In (ii), the limit is uniform in proper subsectors of the upper and lower half-planes

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# NLS: Inverse Scattering Map

We can now define the *inverse scattering map*  $\mathcal{I} : r \mapsto q$  as follows.

**Step 1**: Given  $r \in H_1^{1,1}(\mathbb{R})$ , solve the Riemann-Hilbert problem:

Find  $\mathbf{M}(x, z)$  analytic in  $z \in \mathbb{C} \setminus \mathbb{R}$  for each x so that

- (i)  $\lim_{z\to\infty} \mathbf{M}(x,z) = \mathbb{I}$
- (ii)  $\mathbf{M}(x, z)$  has continuous boundary values  $\mathbf{M}_{\pm}(x, \lambda)$  on  $\mathbb{R}$

(iii) The jump relation

$$\mathbf{M}_{+}(x,\lambda) = \mathbf{M}_{-}(x,\lambda)\mathbf{V}(x,\lambda), \quad \mathbf{V}(x,\lambda) = \begin{pmatrix} 1 - |r(\lambda)|^{2} & -\overline{r(\lambda)}e^{-2i\lambda x} \\ r(\lambda)e^{2i\lambda x} & 1 \end{pmatrix}$$

holds

**Step 2**: Recover q(x) from the reconstruction formula

$$q(x) = \lim_{z \to \infty} 2iz \mathbf{M}_{12}(x, z)$$

#### 

### NLS: Inverse Scattering Map

Deift and Zhou (2003) proved the following mapping property of the inverse scattering map. Recall

$$H^{1,1}(\mathbb{R}) = \{ u \in L^2(\mathbb{R}) : u', xu \in L^2(\mathbb{R}) \}$$
$$H^{1,1}_1(\mathbb{R}) = \{ u \in H^{1,1}(\mathbb{R}) : ||u||_{L^{\infty}} < 1 \}$$

#### Theorem (Deift-Zhou)

*The inverse scattering map*  $\mathcal{I} : r \to q$  *is a Lipschitz continuous map from*  $H_1^{1,1}(\mathbb{R})$  *onto*  $H^{1,1}(\mathbb{R})$ .

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### NLS: Reconstruction Formula

Inverse Scattering

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Combining the three results:

- The direct scattering map *R* : *q* → *r* is a Lipschitz map from *H*<sup>1,1</sup>(**ℝ**) onto *H*<sup>1,1</sup><sub>1</sub>(**ℝ**)
- (2) The reflection coefficient evolves according to r(λ, t) = e<sup>2iλ<sup>2</sup>t</sup>r(λ, 0), a continuous curve in H<sup>1,1</sup>(ℝ)
- (3) The inverse scattering map *I* : r → q is a Lipschitz map from H<sub>1</sub><sup>1,1</sup>(ℝ) onto H<sup>1,1</sup>(ℝ)

we obtain the solution formula

$$q(t,x) = \mathcal{I}\left(e^{2i\lambda^2 t}\mathcal{R}(q_0)\right)(x)$$

which defines a continuous map

$$H^{1,1}(\mathbb{R}) \times \mathbb{R} \ni (q_0, t) \mapsto H^{1,1}(\mathbb{R})$$

### **NLS:** Perturbations

Inverse Scattering

Another important application of the inverse scattering map is to perturbed defocussing NLS equations (Deift-Zhou 2002)

$$\begin{cases} iq_t + q_{xx} - 2|q|^2 q - \varepsilon |q|^l q = 0\\ q(0, x) = q_0(x) \end{cases}$$

for  $\varepsilon > 0$  and l > 2. Here one can show that, for a solution of the perturbed equation, the reflection coefficient  $\mathcal{R}(q(t))$  obeys a 'Duhamel' type equation

$$\begin{cases} \partial_t r(z;q(t)) = -iz^2 r(z;q(t)) + \varepsilon \int_{\mathbb{R}} e^{-iyz} (m_-^{-1}Gm_-)_{12} \, dy \\ r(0) = \mathcal{R}(q_0) \end{cases}$$

where  $m_{-}$  is a scattering solution for q(t) and

$$G = G(q) = -i|q|^l \begin{pmatrix} 0 & q \\ -\overline{q} & 0 \end{pmatrix}$$

Continuity properties of the scattering map  $\mathcal{R}$  are crucial to solving this equation and constructing a solution of the perturbed NLS equation by inverse scattering methods.



We'll give key ideas of the proof that

$$\mathcal{R}: H^{1,1}(\mathbb{R}) \ni q \to r \in H^{1,1}_1(\mathbb{R})$$

is Lipschitz continuous. Recall that  $\Psi^{\pm}(x, \lambda)$  solve

$$\frac{\partial \Psi^+}{\partial x} = i\lambda\sigma_3\Psi + \mathbf{Q}\Psi, \quad \mathbf{Q} = \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix}$$

and

$$\Psi^{+}(x,\lambda) = \Psi^{-}(x,\lambda) \begin{pmatrix} a(\lambda) & \overline{b(\lambda)} \\ b(\lambda) & \overline{a(\lambda)} \end{pmatrix}$$

with  $r(\lambda) = -b(\lambda)/\overline{a(\lambda)}$ .

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To study  $\mathcal{R} : q \to r$ , set

$$\Psi^+(x,\lambda) = e^{-i\lambda x\sigma_3} \mathbf{N}(x,\lambda)$$

which satisfies

$$\begin{cases} \frac{\partial}{\partial x} \mathbf{N}(x,\lambda) = \begin{pmatrix} 0 & e^{2i\lambda x} q(x) \\ e^{-2i\lambda x} \overline{q(x)} & 0 \end{pmatrix} \mathbf{N}(x,\lambda), \\ \lim_{x \to +\infty} \mathbf{N}(x,\lambda) = \mathbb{I}. \end{cases}$$
(3)

and note that

 $\lim_{x\to-\infty}\mathbf{N}(x,\lambda)=T(\lambda).$ 



The function N obeys the integral equation

$$\mathbf{N}(x,\lambda) = \mathbb{I} - \int_x^\infty \begin{pmatrix} 0 & e^{2i\lambda y} q(y) \\ e^{-2i\lambda y} \overline{q(y)} & 0 \end{pmatrix} \mathbf{N}(y,\lambda) \, dy$$

and

$$\lim_{x \to -\infty} \mathbf{N}(x, \lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ b(\lambda) & \overline{a(\lambda)} \end{pmatrix}$$

Focus on

$$\begin{pmatrix} N_{11}(x,\lambda)\\ N_{21}(x,\lambda) \end{pmatrix} = \begin{pmatrix} 1\\ 0 \end{pmatrix} - \int_x^\infty \begin{pmatrix} e^{2i\lambda y}q(y)N_{21}(y,\lambda)\\ e^{-2i\lambda y}\overline{q(y)}N_{11}(y,\lambda) \end{pmatrix} \, dy$$

which is a Volterra-type integral equation solvable by a Volterra series

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# Lipschitz Continuity of $\mathcal{R}$

Renaming  $N_{11}(x,\lambda) = a(x,\lambda)$ ,  $N_{21}(x,\lambda) = b(x,\lambda)$ , we have

$$a(x,\lambda) = 1 + \sum_{n=1}^{\infty} A_{2n}(x,\lambda), \qquad b(x,\lambda) = -\sum_{n=0}^{\infty} A_{2n+1}(x,\lambda)$$

and on taking limits

$$a(\lambda) = 1 + \sum_{n=1}^{\infty} A_{2n}(\lambda), \qquad \qquad b(\lambda) = -\sum_{n=0}^{\infty} A_{2n+1}(\lambda)$$

where

$$A_n(\lambda) = \int_{y_1 < y_2 < \ldots < y_n} Q_n(y_1, \ldots, y_n) e^{2i\lambda \phi_n(y_1, \ldots, y_n)} \, dy_n \, \ldots \, dy_1,$$

 $\varphi_n$  is a real phase function, and

$$Q_n(y_1,\ldots,y_n) = \begin{cases} \prod_{j=1}^m q(y_{2j-1})\overline{q(y_{2j})}, & n = 2m, \\ \\ \overline{q(y_1)}\prod_{j=1}^m q(y_{2j})\overline{q(y_{2j+1})}, & n = 2m+1. \end{cases}$$

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We have the multilinear series

$$a(\lambda) = 1 + \sum_{n=1}^{\infty} A_{2n}(\lambda), \qquad b(\lambda) = -\sum_{n=0}^{\infty} A_{2n+1}(\lambda)$$

where  $A_m$  is a multilinear integral with phase function and amplitude  $Q_m$ :

$$Q_n(y_1,\ldots,y_n) = \begin{cases} \prod_{j=1}^m q(y_{2j-1})\overline{q(y_{2j})}, & n=2m, \\ \\ \overline{q(y_1)}\prod_{j=1}^m q(y_{2j})\overline{q(y_{2j+1})}, & n=2m+1. \end{cases}$$

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From these series one can analyze the map  $q \mapsto r$  in four stages:

- (1) Show that  $\mathcal{R}: L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$
- (2) Show that  $\mathcal{R}: L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \to L^2(\mathbb{R})$
- (3) Show that  $\mathcal{R}: H^{1,1}(\mathbb{R}) \to H^{0,1}(\mathbb{R})$
- (4) Show that  $\mathcal{R}: H^{1,1}(\mathbb{R}) \to H^{1,1}(\mathbb{R})$



To prove Lipschitz continuity of  $\mathcal{I} : r \to q$ , we need to:

- 1. Reduce the Riemann-Hilbert problem to an integral equation, the *Beals-Coifman integral equation*
- 2. Show that the Beals-Coifman integral equation admits a unique solution  $\mu$  for  $r \in H_1^{1,1}(\mathbb{R})$
- 3. Find an explicit reconstruction formula in terms of *r* and  $\mu$  and use it to establish Lipschitz continuity of  $\mathcal{I}$

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*Remark*: Recall the  $\overline{\partial}$  operator for z = x + iy:

$$\overline{\partial} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

and that  $\overline{\partial}F = 0$  if *F* is analytic.

Our Riemann-Hilbert problem can be viewed as a  $\overline{\partial}$  problem with boundary conditions:

$$\begin{split} &(\overline{\partial}\mathbf{M})(z) = 0 & z \in \mathbb{C} \setminus \mathbb{R}, \\ &\mathbf{M}_{+}(z) = \mathbf{M}_{-}(z) \begin{pmatrix} 1 - |r(\lambda)|^2 & \overline{r(\lambda)}e^{-i\lambda x} \\ -r(\lambda)e^{i\lambda x} & 1 \end{pmatrix}, z \in \mathbb{R} \end{split}$$

It is natural that a PDE boundary value problem can be reduced to a boundary integral equation

# **Beals-Coifman Integral Equation**

We can write the jump relation for **M** in the form

$$\mathbf{M}_{+}(z) = \mathbf{M}_{-}(z)(I - w_{x}^{-}(\lambda))^{-1}(I + w_{x}^{+}(\lambda))$$

where

$$w_x^-(\lambda) = \begin{pmatrix} 0 & 0 \\ e^{2i\lambda x}r(\lambda) & 0 \end{pmatrix}, \quad w_x^+(\lambda) = \begin{pmatrix} 0 & -e^{2i\lambda x}\overline{r(\lambda)} \\ 0 & 0 \end{pmatrix}$$

Now let

$$\mu(x,\lambda) = \mathbf{M}_+(x,\lambda)(I+w_x^+(\lambda))^{-1} = \mathbf{M}_-(x,\lambda)(I-w_x^-(\lambda))^{-1}$$

Then

$$\mathbf{M}_{+}(x,\lambda) - \mathbf{M}_{-}(x,\lambda) = \mu(x,\lambda)(w_{x}^{+}(\lambda) + w_{x}^{-}(\lambda))$$

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Scattering Maps



# **Beals-Coifman Integral Equation**

#### Recall

$$\mu(x,\lambda) = \mathbf{M}_{+}(x,\lambda)(I + w_{x}^{+}(\lambda))^{-1} = \mathbf{M}_{-}(x,\lambda)(I - w_{x}^{-}(\lambda))^{-1}$$
(4)

and

$$\mathbf{M}_{+}(x,\lambda) - \mathbf{M}_{-}(x,\lambda) = \mu(x,\lambda)(w_{x}^{+}(\lambda) + w_{x}^{-}(\lambda))$$

From the formula

$$\mathbf{M}(x,z) = \mathbb{I} + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\mathbf{M}_{+}(x,\lambda) - \mathbf{M}_{-}(x,\lambda)}{\lambda - z} \, d\lambda$$

we get

$$\mathbf{M}(x,z) = \mathbb{I} + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\mu(x,\lambda)(w_x^+(\lambda) + w_x^-(\lambda))}{\lambda - z} \, d\lambda \tag{5}$$

We will use (4) and (5) to derive the Beals-Coifman integral equation.

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# **Beals-Coifman Integral Equation**

Recall that  $\mu(x, \lambda)(I + w_x^+(\lambda)) = \mathbf{M}_+(x, \lambda)$ .

For  $f \in H^1(\mathbb{R})$ , define the *Cauchy projectors*  $C_{\pm}$  by

$$(C_{\pm}f)(\lambda) = \lim_{\epsilon \downarrow 0} \int \frac{f(s)}{s - (\lambda \pm i\epsilon)} \, ds,$$

and recall that  $\|C_{\pm}\|_{L^2 \to L^2} = 1$  and  $C_+ - C_- = I$ . Using

$$\mathbf{M}(x,z) = \mathbb{I} + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\mu(x,\lambda)(w_x^+(\lambda) + w_x^-(\lambda))}{\lambda - z} \, d\lambda$$

we take boundary values to recover

$$\mathbf{M}_+(x,\lambda) = \mathbb{I} + C_+(\mu w_x^+ + \mu w_x^-).$$

Hence

$$\mu(x,\lambda)(I+w_x^+(\lambda)) = \mathbb{I} + C_+(\mu w_x^+ + \mu w_x^-)$$

or

$$\mu = \mathbb{I} + C_{+}(\mu w_{x}^{-}) + C_{-}(\mu w_{x}^{+})$$



# **Beals-Coifman Integral Equations**

Let

$$\mathcal{C}_w(h) = C_+(hw_x^-) + C_-(hw_x^+)$$

The equation

$$\mu = \mathbb{I} + \mathcal{C}_w(\mu) \tag{6}$$

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is the Beals-Coifman Integral Equation.

**Theorem** Suppose that  $r \in H^{1,1}(\mathbb{R})$  with  $||r||_{L^{\infty}} = \rho < 1$ . There exists a unique solution  $\mu$  of (6) with  $\mu(\lambda) - 1 \in L^2(\mathbb{R})$ .

This is a consequence of the operator estimate

$$\|\mathcal{C}_w\|_{L^2 \to L^2} = \|r\|_{L^{\infty}}$$

and the solution formula

$$\mu - \mathbb{I} = (I - \mathcal{C}_w)^{-1} \mathcal{C}_w(\mathbb{I})$$

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# Vanishing Theorem

Using the reduction of the RHP to an integral equation, we can also prove a uniqueness theorem for the RHP.

### Theorem (Vanishing Theorem)

*Fix*  $x \in \mathbb{R}$ , suppose  $r \in H^{1,0}(\mathbb{R})$  with  $||r||_{L^{\infty}} < 1$  and suppose that  $\mathbf{n}(x,z) : \mathbb{C} \setminus \mathbb{R} \to M_2(\mathbb{C})$  solves the Riemann-Hilbert problem with boundary values  $\mathbf{n}_{\pm}(x,z) \in L^2(\mathbb{R})$ . Then  $\mathbf{n}(x,z) \equiv 0$ .

Idea of proof: Repeat the reduction to a Beals-Coifman integral equation with

$$\nu(x,\lambda) = \mathbf{n}_+(x,\lambda)(I + w_x^+(\lambda))^{-1} = \mathbf{n}_-(x,\lambda)(I - w_x^-(\lambda))^{-1}$$

and arrive at the integral equation

$$\nu = C_w \nu$$

which has only the zero solution.

A vector **n** satisfying the hypothesis of the vanishing theorem is called a *null vector* for the RHP.



#### **Reconstruction Formula**

The solution  $\mu$  of the Beals-Coifman integral equation determines the solution **M** of the Riemann-Hilbert problem via

$$\mathbf{M}(x,z) = \mathbb{I} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\mu(x,\lambda)(w_x^+(\lambda) + w_x^-(\lambda))}{\lambda - z} \, d\lambda$$

#### Theorem

Suppose that  $r \in H_1^{1,1}(\mathbb{R})$ , let  $\mathbf{M}(x,z)$  be the unique solution of the Riemann-Hilbert problem. Then

$$\frac{d}{dx}\mathbf{M}(x,z) = -iz \operatorname{ad} \sigma_3(\mathbf{M}) + \mathbf{Q}(x)\mathbf{M}(x,z)$$

where

$$\mathbf{Q}(x) = \begin{pmatrix} 0 & q(x) \\ \overline{q(x)} & 0 \end{pmatrix}, \qquad q(x) = -\frac{1}{\pi} \int \overline{r(s)} e^{-2ixs} \mu_{11}(x,s) \, ds$$

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#### Idea of Proof

To prove that

$$\frac{d}{dx}\mathbf{M}(x,z) = -iz \operatorname{ad} \sigma_3(\mathbf{M}) + \mathbf{Q}(x)\mathbf{M}(x,z)$$

and identify  ${\bf Q},$  differentiate the jump relation for  ${\bf M}$  to obtain

$$\left(\frac{\partial \mathbf{M}_{+}}{\partial x} + i\lambda \operatorname{ad} \sigma_{3}(\mathbf{M}_{+})\right) = \left(\frac{\partial \mathbf{M}_{-}}{\partial x} + i\lambda \operatorname{ad} \sigma_{3}(\mathbf{M}_{-})\right) \mathbf{V}(x,\lambda)$$

and show that

$$i\lambda$$
 ad  $\sigma_3(\mathbf{M}_{\pm}) - \mathbf{Q}(x) \in L^2(\mathbb{R})$ 

to conclude that

$$\mathbf{n}(x,z) = \frac{\partial}{\partial x} \mathbf{M}(x,z) + i\lambda \operatorname{ad} \sigma_3(\mathbf{M}(x,z)) - \mathbf{Q}(x)\mathbf{M}(x,z)$$

is a null vector for the Riemann-Hilbert problem, hence  $\mathbf{n}(x, z) \equiv 0$ .

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From

$$q(x) = -\frac{1}{\pi} \int \overline{r(s)} e^{-2ixs} \mu_{11}(x,s) \, ds$$

write  $q(x) = q_0(x) + q_1(x)$  where

$$q_0(x) = -\frac{1}{\pi} \int \overline{r(s)} e^{-2ixs} ds$$
  
$$q_1(x) = -\frac{1}{\pi} \int \overline{r(s)} e^{-2ixs} (\mu_{11}(x,s) - 1) ds$$

The map  $r \mapsto q_0$  is a Fourier transform with the required properties. To analyze  $r \mapsto q_1$  we use the identity

$$q_1'(x) = -q(x) \left(\frac{1}{\pi} \int \overline{r(s)} e^{-2ixs} \mu_{21}(x,s) \, ds\right)$$

and use Lipschitz continuity properties of  $r \mapsto \mu_{11} - 1$  and  $r \mapsto \mu_{12}$ .