Inverse Scattering in Two Dimensions: The Davey-Stewartson II Equation

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The Davey-Stewartson II Equation

The Davey-Stewartson II (DS II) equation is a nonlinear dispersive equation in *two* space dimensions. The Cauchy problem is

$$\begin{cases} iq_t + 2(\partial_z^2 + \partial_{\overline{z}}^2)q + (g + \overline{g})q = 0 \\ \partial_{\overline{z}} g + 4\varepsilon \partial_z(|q|^2) = 0 \\ q(0,z) = q_0(z) \end{cases}$$

where $\varepsilon = \pm 1$, $z = x_1 + ix_2$ and

$$\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \partial_{\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right),$$

The case $\varepsilon = 1$ is the *defocussing* case (no solitons), and the case $\varepsilon = -1$ is the *focussing* case (soliton solutions and blow-up in finite time). In what follows, the notation $f(z)$ for a function of $z = x_1 + ix_2$ does *not* imply that f is analytic.

Ablowitz and Haberman (1975) showed that the DS II equation is completely integrable.

Local Well-Posedness Theory for DS II

Ghidaglia and Saut (*Nonlinearity* **3** (1990), no. 2, 475-506) proved:

Theorem

For any $q_0 \in L^2(\mathbb{R}^2)$*, there is a* $T^* > 0$ *and a unique solution* $q(t)$ *to the DS II equation belonging to* $C((0,T^*), L^2(\mathbb{R}^2)) \cap L^4(\mathbb{R}^2 \times (0,T^*))$ *with* $q(t) = q_0$ *and* $\|q(t)\|_{L^2} = \|q_0\|_{L^2}$.

This result applies to the DS II equation with either $\varepsilon = 1$ or $\varepsilon = -1$. These authors considered a class of equations which includes the completely integrable equations as a special case.

For the focussing DS II equation, there is a solution to the focussing DS II equation which blows up in finite time (Ozawa, *Proc. Roy. Soc. London* **436** (1992)).

Overview - Inverse Scattering

For the *defocussing* DS II equation, global existence in $L^2(\mathbb{R}^2)$ and nonlinear scattering are now established by inverse scattering methods (Nachman, Regev, Tataru (2020)). Pointwise asymptotics are also known (Kiselev 1997, Perry 2016).The defocussing DS II equations does not have soliton solutions in the usual sense.

For the *focussing* DS II equation, there are only small-data results for inverse scattering due to the possibility of blow-up in finite time and soliton solutions for which the inverse scattering theory is not rigorously established (Sung 1994). Pointwise asymptotics for small data are known (Kiselev 1997)

For the focussing DS II equation, it is also known that the one-soliton solution is spectrally unstable (Brown, Perry 2018)

First Result

Theorem (Perry 2014)

Suppose that $q_0 \in H^{1,1}(\mathbb{R}^2)$. The Davey-Stewartson II equation has a global solution $q(t,z)$ with $q(t,\,\cdot\,) \in H^{1,1}(\mathbb{R}^2)$, and the map

$$(t, q_0) \mapsto q(t,\,\cdot\,)$$
$$\mathbb{R} \times H^{1,1}(\mathbb{R}^2) \to C(\mathbb{R}, H^{1,1}(\mathbb{R}^2))$$

is locally Lipschitz continuous. Moreover, if $q_0 \in H^{1,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ with $(\mathcal{S}q_0)(0) = 0$, then

$$q(t,z) = v(t,z) + o(t^{-1})$$

where $v(t,z)$ solves the problem

$$i\partial_t v + 2(\partial_z^2 + \partial_{\bar{z}}^2)v = 0$$

with initial data

$$v(0,z) = \mathcal{F}_a(\mathcal{S}(q_0)).$$

Second Result

Theorem (Nachman, Regev, Tataru 2020)

Suppose that $q_0 \in L^2(\mathbb{R}^2)$. The Davey-Stewartson II equation has a unique global solution $q(t,z) \in C(\mathbb{R}, L^2(\mathbb{C})) \cap L^4_{t,z}(\mathbb{R} \times \mathbb{C})$ with

(i) $\|q(0)\|_{L^2(\mathbb{C})} = \|q(t)\|_{L^2(\mathbb{C})}$ *and* $\int_{\mathbb{R}\times\mathbb{C}} |q(t,z)|^4 \, dt \, dz \leq C(\|q_0\|_{L^2(\mathbb{C})})$

(ii) *If $q_1(t,\cdot)$ and $q_2(t,\cdot)$ are solutions corresponding to initial data q_1 and q_2 with $\|q_i\|_{L^2(\mathbb{C})} \leq R$, $i = 1, 2$, then*

$$\|q_1(t,\cdot) - q_2(t,\cdot)\|_{L^2(\mathbb{C})} \leq C(R) \, \|q_1 - q_2\|_{L^2(\mathbb{C})}.$$

We will discuss Perry's result in this lecture, using ideas of Nachman, Regev, and Tataru to simplify the proofs. We will discuss Nachman, Regev, and Tataru's result in more detail in the next lecture.

Lax Representation

The DS II equation is the compatibility condition for the following system of equations for unknowns $\psi_1(z,t,k)$ and $\psi_2(z,t,k)$:

$$\begin{aligned}\partial_{\bar{z}}\psi_1 &= q\psi_2\\ \partial_z\psi_2 &= \varepsilon\bar{q}\psi_1\end{aligned}$$

$$\begin{aligned}\partial_t\psi_1 &= \quad 2i\partial_z^2\psi_1 + 2i(\partial_{\bar{z}}q)\psi_2 - 2iq\partial_{\bar{z}}\psi_2 + g\psi_1\\ \partial_t\psi_2 &= -2i\partial_{\bar{z}}^2\psi_2 - 2i\varepsilon(\partial_z q)\psi_1 + 2i\varepsilon\bar{q}\partial_z\psi_1 - i\bar{g}\psi_2\end{aligned}$$

One can check that DS II is the compatibility condition by cross-differentiating in x and t.

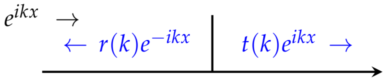
The first system of equations defines a *scattering transform*

$$\mathcal{S} : q \to \mathbf{s},$$

and the second system of equations determines the time evolution of scattering data. We will mainly discuss the defocussing ($\varepsilon = +1$) case

Interlude: Dimension Counting

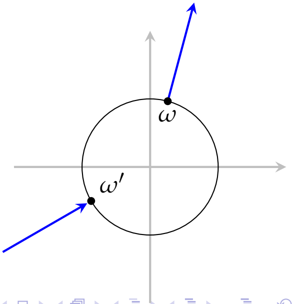
In one space dimension, the scattering data is a scalar function and $\mathcal{S} : q \to r$



In two space dimensions, scattering data is a function

$$S(\omega, \omega'; E), \quad \omega, \omega' \in S^1 \text{ and } E \in \mathbb{R}$$

which actually depends on *three variables*! Thus, in two dimensions, it is more natural to consider scattering (and inverse scattering) at *fixed energy* to get a map $\mathcal{S} : q \to \mathbf{s}$



Direct Scattering

The defocusing DS II flow is linearized by a *zero-energy* spectral problem for the operator

$$\mathcal{L} = \begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\overline{z}} \end{pmatrix} - \mathbf{Q}(z), \quad \mathbf{Q}(z) = \begin{pmatrix} 0 & q(z) \\ \overline{q(z)} & 0 \end{pmatrix}$$

To find the scattering transform for $q \in \mathcal{S}(\mathbb{R}^2)$, look for solutions

$$\begin{pmatrix} \psi_1(z,k) \\ \psi_2(z,k) \end{pmatrix} = \begin{pmatrix} m^1(z,k)e^{ikz} \\ m^2(z,k)e^{ikz} \end{pmatrix}, \quad \lim_{|z|\to\infty} (m^1(z,k), m^2(z,k)) = (1,0).$$

of $\mathcal{L}\psi = \mathbf{0}$ where $k = k_1 + ik_2$, $z = x_1 + ix_2$, and kz denotes complex multiplication. One can check that

$$\begin{cases} \partial_{\overline{z}} m^1(z,k) = q(z) m^2(z,k) \\ (\partial_z + ik) m^2(z,k) = \overline{q(z)} m^1(z,k) \\ \lim_{|z|\to\infty} (m^1(z,k), m^2(z,k) = (1,0) \end{cases}$$

Direct Scattering

The scattering problem for m^1 and m^2 is equivalent to

$$\begin{cases} m^1(z,k) = 1 + \dfrac{1}{\pi}\displaystyle\int_{\mathbb{C}} \dfrac{1}{z-w} q(w) m^2(w,k)\, dw \\ m^2(z,k) = \dfrac{1}{\pi}\displaystyle\int_{\mathbb{C}} \dfrac{e_k(z-w))}{\overline{z}-\overline{w}} \overline{q(w)} m^1(z,w)\, dw \end{cases}$$

where $e_k(z) = e^{i(kz+\overline{k}\overline{z})}$. For $q \in \mathcal{S}(\mathbb{R}^2)$, we have

$$m^1(z,k) \underset{z\to\infty}{\sim} 1 + \sum_{j\geq 1} \frac{a_j(k)}{z^j}, \qquad m^2(z,k) \underset{z\to\infty}{\sim} e_{-k}(z) \sum_{j\geq 1} \frac{b_j(k)}{\overline{z}^j}$$

The scattering transform is $(\mathcal{S}q)(k) = -ib_1(k)$ or, from the integral equations

$$(\mathcal{S}q)(k) = -\frac{i}{\pi}\int_{\mathbb{C}} e_k(z)\overline{q(z)} m^1(z,k)\, dz.$$

which is perturbation of the antilinear 'Fourier transform'

$$(\mathcal{F}_a q)(k) = -\frac{i}{\pi}\int_{\mathbb{C}} e_k(z)\overline{q(z)}\, dz.$$

Inverse Scattering

Let $\mathbf{s}(k) = \mathcal{S}(q)(k)$. If $q \in \mathcal{S}(\mathbb{R}^2)$, it follows from the work of Beals and Coifman that $\mathbf{s} \in \mathcal{S}(\mathbb{R}^2)$.

The scattering solutions m^1, m^2 also obey the dual set of equations

$$\begin{aligned} \partial_{\overline{k}} m^1(z,k) &= e_{-k}(z)\mathbf{s}(k)\overline{m^2(z,k)} \\ \partial_{\overline{k}} m^2(z,k) &= e_{-k}(z)\mathbf{s}(k)\overline{m^1(z,k)} \\ \lim_{|k|\to\infty} (m^1(z,k), m^2(z,k)) &= (1,0). \end{aligned}$$

With the change of variables $n^1 = m^1$, $n^2 = e_{-k}\overline{m^2}$ we obtain

$$\begin{aligned} \partial_{\overline{k}} n^1(z,k) &= \mathbf{s}(k) n^2(z,k) \\ (\partial_k + iz) n^2(z,k) &= \overline{\mathbf{s}(k)} n^1(z,k) \\ \lim_{|k|\to\infty} (n^1(z,k), n^2(z,k)) &= (1,0) \end{aligned}$$

which have the same form as the z-equations with the roles of z and k reversed.

Inverse Scattering

For the system

$$\partial_{\overline{k}} n^1(z,k) = \mathbf{s}(k) n^2(z,k)$$
$$(\partial_k + iz) n^2(z,k) = \overline{\mathbf{s}(k)} n^1(z,k)$$
$$\lim_{|k|\to\infty} (n^1(z,k), n^2(z,k)) = (1,0)$$

we have

$$n^2(z,k) \underset{k\to\infty}{\sim} e_{-k}(z) \frac{iq(z)}{\overline{k}} + \mathcal{O}\left(|k|^{-2}\right)$$

or

$$q(z) = -\frac{i}{\pi} \int e_k(z) \overline{\mathbf{s}(k)} n^1(z,k)\, dk$$

Time Evolution of Scattering Data

Suppose that $q(t,z)$ solves the DS II equation with $q(t,\cdot\,) \in \mathcal{S}(\mathbb{R}^2)$ for each t (this is true if $q(0,\cdot\,) \in \mathcal{S}(\mathbb{R}^2)$). We can determine the evolution of scattering data as follows.

(1) Find a law of evolution for solutions of the Lax equations

(2) Use the large-z asymptotics for the functions m^1 and m^2 to recover the scattering data

(3) Use the Lax equations for time evolution together with the asymptotics of m^1 and m^2 to find equations of motion for the scattering data

Time Evolution of Scattering Data (1 of 3)

Step 1: Find the law of evolution for solutions of the Lax equations.

Let

$$\psi_1 = C(k,t)e^{ikz}m^1, \quad \psi_2 = C_2(k,t)e^{ikz}m^2$$

where, for each t,

$$(m^1(z,k,t), m^2(z,k,t)) \underset{|z|\to\infty}{\sim} (1,0)$$

Substitute into the second pair of Lax equations and use large-z asymptotics to recover

$$C_1(k,t) = C_2(k,t) = e^{2it(k^2+\overline{k}^2)}.$$

In the m_1, m_2 variables, the second pair of Lax equations is now

$$\partial_t m^1 = 2i(\partial_z^2 + 2ik\partial_z)m^1 + 2i(\partial_{\overline{z}} q)m^2 - 2iq\partial_{\overline{z}} m^2 + igm^1$$
$$\partial_t m^2 = -2i\partial_{\overline{z}}^2 m^2 + 2ik^2 m^2 - 2i\varepsilon(\partial_z q)m^1 + 2i\varepsilon(\partial_z + ik)m^1 - i\overline{g}m^2$$

Time Evolution of Scattering Data (2 of 3)

Step 2: Use the large-z asymptotics of m^1 and m^2 to recover the scattering data

If $q(t, \cdot) \in \mathcal{S}(\mathbb{R}^2)$, m^1 and m^2 should have large-z expansions

$$m^1(z,k,t) \underset{|z|\to\infty}{\sim} 1 + \frac{a(k,t)}{z} + \mathcal{O}\left(|z|^{-2}\right)$$

$$m^2(z,k,t) \underset{|z|\to\infty}{\sim} e_{-k}(z)\frac{b(k,t)}{\overline{z}} + \mathcal{O}\left(|z|^{-2}\right)$$

where $b(k,t) = i\mathbf{s}(k,t)$.

Time Evolution of Scattering Data (3 of 3)

Step 3: Use the Lax equations to recover time evolution of scattering data
Substitute

$$m^1(z,k,t) \underset{|z|\to\infty}{\sim} 1 + \frac{a(k,t)}{z} + \mathcal{O}\left(|z|^{-2}\right)$$
$$m^2(z,k,t) \underset{|z|\to\infty}{\sim} e_{-k}(z)\frac{b(k,t)}{\overline{z}} + \mathcal{O}\left(|z|^{-2}\right),$$

where

$$b(k,t) = i\mathbf{s}(k,t),$$

into

$$\partial_t m^1 = 2i(\partial_z^2 + 2ik\partial_z)m^1 + 2i(\partial_{\overline{z}}q)m^2 - 2iq\partial_{\overline{z}}m^2 + igm^1$$
$$\partial_t m^2 = -2i\partial_{\overline{z}}^2 m^2 + 2ik^2 m^2 - 2i\varepsilon(\partial_z q)m^1 + 2i\varepsilon(\partial_z + ik)m^1 - i\overline{g}m^2$$

to recover

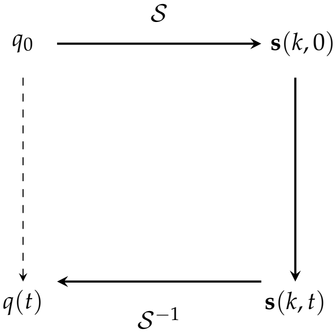
$$\dot{a}(k,t) = 0, \quad \dot{b}(k,t) = 2i(k^2 + \overline{k}^2)b(k,t)$$

Reconstruction Formula

We now have

$$\mathbf{s}(k,t) = e^{2i(k^2+\bar{k}^2)}\mathbf{s}(k,0)$$

which gives the following reconstruction formula: if $\mathbf{s}(k,0) = (\mathcal{S}q_0)(k)$,



We will show that the maps $\mathcal{S}$ and $\mathcal{S}^{-1} = \mathcal{S}$ are well-defined and Lipschitz continuous

Continuity of the Scattering Map

Let

$$H^{1,1}(\mathbb{R}^2) = \left\{ u \in L^2(\mathbb{R}^2) : \nabla u,\ |x|u \in L^2(\mathbb{R}^2) \right\}$$

with norm

$$\|u\|_{H^{1,1}(\mathbb{R}^2)} = \|(1+|x|)u\|_{L^2} + \|\nabla u\|_{L^2}$$

We will show that

$$\mathcal{S} : H^{1,1}(\mathbb{R}^2) \to H^{1,1}(\mathbb{R}^2)$$

is one-to-one, onto, and locally Lipschitz. This will imply that the map

$$\begin{gathered} (t, q_0) \mapsto \mathcal{S}(e^{2it((\cdot)^2 + (\bar{\cdot})^2)} \mathcal{S}(q_0)) \\ \mathbb{R} \times H^{1,1}(\mathbb{R}) \to C(\mathbb{R}, H^{1,1}(\mathbb{R})) \end{gathered}$$

is a locally Lipschitz continuous map, giving global well-posedness in $H^{1,1}(\mathbb{R})$.

Continuity of the Scattering Map

We will show continuity of the scattering map in the following steps:

(1) We will show that the direct scattering problem is uniquely solvable and that the map

$$q \mapsto (m^1 - 1, m^2)$$
$$H^{1,1}(\mathbb{R}^2) \to L^4(\mathbb{R}^2_z \times \mathbb{R}^2_k)$$

is locally Lipschitz continuous

(2) We will show that the scattering map

$$q \mapsto \mathbf{s}(k) = -\frac{i}{\pi}\int_{\mathbb{C}} e_k(z)\overline{q(z)}m^1(z,k)\,dz$$
$$H^{1,1}(\mathbb{R}^2) \to H^{1,1}(\mathbb{R}^2)$$

is Lipschitz continuous

Noting that the map $\mathbf{s}(k) \mapsto e^{2it(k^2+\overline{k}^2)}\mathbf{s}(k)$ is continuous on $H^{1,1}(\mathbb{R}^2)$ the continuity of $q_0 \mapsto \mathcal{S}^{-1}\left(e^{2it((\,\cdot\,)^2+(\overline{\cdot})^2)}\mathcal{S}q_0\right)$ will follow

The Direct Scattering Problem

Recall the problem

$$\begin{cases} \partial_{\bar{z}} m^1(z,k) = q(z) m^2(z,k) \\ (\partial_z + ik) m^2(z,k) = \overline{q(z)} m^1(z,k) \\ (m^1(z,k) - 1, m^2(z,k)) \in L^4_z(\mathbb{R}^2) \end{cases}$$

Let

$$m^{\pm}(z,k) = m^1(z,k) \pm e_{-k}(z) \overline{m^2(z,k)}$$

Then

$$\begin{cases} \partial_{\bar{z}} m^{\pm}(z,k) = \pm e_{-k}(z) q(z) \overline{m^{\pm}(z,k)} \\ m^{\pm}(z,k) - 1 \in L^4_z(\mathbb{R}^2) \end{cases}$$

We will show that solutions exist and are unique.

Uniqueness of Solutions

Recall

$$\begin{cases} \partial_{\bar z} m^{\pm}(z,k) = \pm e_{-k}(z) q(z) \overline{m^{\pm}(z,k)} \\ m^{\pm}(z,k) - 1 \in L^4_z(\mathbb{R}^2) \end{cases}$$

Theorem (Vekua)

Suppose $a \in L^2(\mathbb{R}^2)$, $u \in L^p(\mathbb{R}^2)$ for some $p > 2$, and $\partial_{\bar z} u = a\bar u$ in distribution sense. Then $u \equiv 0$.

Proof.

Let m be a difference of two solutions and apply Vekua's Theorem to the equation

$$\partial_{\bar z} m = \pm e_{-k} q(z) \overline{m}, \quad m \in L^4(\mathbb{R}^2)$$

□

Existence of Solutions

Focus on m^+. Rewrite

$$\begin{cases} \partial_{\overline{z}} m^+(z,k) = e_{-k}(z) q(z) \overline{m^+(z,k)} \\ m^+(z,k) - 1 \in L^4_z(\mathbb{R}^2) \end{cases}$$

as an integral equation for $w = m^+ - 1 \in L^4_z(\mathbb{R}^2)$:

$$\begin{cases} w - Tw = \partial_{\overline{z}}^{-1}(e_{-k} q) \\ \qquad Tf := \partial_{\overline{z}}^{-1}(e_{-k} q \overline{f}) \end{cases}$$

where

$$(\partial_{\overline{z}}^{-1} f)(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(w)}{z - w}\, dw$$

is the Cauchy integral operator.

Properties of the Cauchy Integral Operator

$$(\partial_{\overline{z}}^{-1} f)(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(w)}{z-w}\, dw$$

(1) If $p \in (1,2)$ and p^* satisfies $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{2}$, then

$$\left\| \partial_{\overline{z}}^{-1} f \right\|_{L^{p^*}} \lesssim_p \|f\|_{L^p}$$

(2) For $1 < p < 2 < r < \infty$,

$$\left\| \partial_{\overline{z}}^{-1} f \right\|_{L^\infty} \lesssim_{p,r} \|f\|_{L^p \cap L^r} .$$

(3) For $p \in (2, \infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $\partial_{\overline{z}}^{-1} f$ is Hölder continuous of order $1 - 2/p$. Moreover, if $f \in L^p \cap L^{p'}$ then

$$\lim_{|z| \to \infty} (\partial_{\overline{z}}^{-1} f)(z) = 0$$

Existence of Solutions

Recall $w = m^+ - 1, \quad q \in H^{1,1}(\mathbb{R}^2)$, and

$$\begin{cases} w - Tw = \partial_{\overline{z}}^{-1}(e_{-k}q) \\ \qquad Tf := \partial_{\overline{z}}^{-1}(e_{-k}q\overline{f}) \end{cases}$$

Recall $\partial_{\overline{z}}^{-1} : L^{\frac{4}{3}}(\mathbb{R}^2) \to L^4(\mathbb{R}^2)$

As $H^{1,1} \subset L^{4/3}$, we have $\partial_{\overline{z}}^{-1}(e_{-k}q) \in L^4(\mathbb{R}^2)$.

Let $u = e_{-k}q$ and let

$$(\mathbf{S}f) = \partial_{\overline{z}}^{-1}(u\overline{f}), \quad u \in L^2(\mathbb{R}^2), \quad f \in L^4(\mathbb{R}^2).$$

Then $\mathbf{S} : L^4(\mathbb{R}^2) \to L^4(\mathbb{R}^2)$ with

$$\|\mathbf{S}\|_{L^4 \to L^4} \lesssim \|u\|_{L^2}$$

So the integral equation makes sense on $L^4(\mathbb{R}^2)$.

Existence of Solutions

$$w - \mathbf{S}w = g \quad \text{where} \quad \mathbf{S}(w) = \partial_{\overline{z}}^{-1}(u\overline{w}), \quad g = \partial_{\overline{z}}^{-1}(u)$$

We claim:

(1) **S** is compact, as follows from the Kolmogorov-Riesz Theorem

(2) $\ker(I - \mathbf{S}) = \{0\}$

To prove (1), show that **S** maps the unit ball in $L^4(\mathbb{R}^2)$ into a compact set.

To prove (2), suppose that $\mathbf{S}f = f$. Then

$$\partial_{\overline{z}} f = e_{-k} q \overline{f}$$

Apply Vekua's Theorem to conclude that $f \equiv 0$.

It now follow from the Fredholm alternative that the integral equation $w - \mathbf{S}w = g$ is uniquely solvable.

Bounds and Continuity

Large-k bounds on the resolvent: Integrate by parts to compute

$$\partial_{\overline{z}}^{-1}(e_{-k}f)(z) = \frac{1}{\pi i \overline{k}}\left(-e_{-k}(z)f(z) + \int \frac{e_{-k}(w)(\partial_{\overline{z}} f)(w)}{z-w}\, dw\right)$$

and deduce

$$(Tf)(z) = \frac{1}{\pi i \overline{k}}\left(-e_{-k}(z)q(z)\overline{f(z)} + \int \frac{e_{-k}(w)\partial_{\overline{z}}(q\overline{f})(w)}{z-w}\, dw\right)$$

so that

$$\|Tf\|_{L^4} \lesssim \frac{1}{|k|}\left(\left\|q\overline{f}\right\|_{L^4} + \left\|\partial_{\overline{z}}(q\overline{f})\right\|_{L^{4/3}}\right)$$

and by iteration

$$\begin{aligned}\left\|T^2 f\right\|_{L^4} &\lesssim \frac{1}{|k|}\left(\|qTf\|_{L^4} + \|\partial_{\overline{z}}(qTf)\|_{L^{4/3}}\right)\\ &\lesssim \frac{1}{|k|}\left(\|q\|_{L^8}\|Tf\|_{L^8} + \|\partial_{\overline{z}} q\|_{L^2}\|Tf\|_{L^4} + \|q\|_{L^4}^2\|f\|_{L^4}\right).\end{aligned}$$

Conclude that

$$\left\|T^2\right\|_{L^4\to L^4} \lesssim \frac{1}{|k|}\|q\|_{H^{1,1}}^2$$

Bounds and Continuity

Lemma

Fix $R > 0$. There is an $N = N(R)$ so that for all $k \in \mathbb{C}$ with $|k| \geq N$, $q \in H^{1,1}(\mathbb{R}^2)$, $\|q\|_{H^{1,1}} < R$,

$$\left\| (I - T)^{-1} \right\|_{L^4 \to L^4} < 2$$

Proof.

Use the bound

$$\left\| T^2 \right\|_{L^4 \to L^4} \lesssim \frac{1}{|k|} \|q\|_{H^{1,1}}^2$$

and the identity

$$(I - T)^{-1} = (I - T^2)^{-1}(I + T)$$

□

To obtain bounded invertibility for $|k| < N$, use a compactness argument:

Bounds and Continuity

Lemma

Suppose B is a bounded subset of $H^{1,1}(\mathbb{C}) \times \mathbb{C}$. Then

$$\sup_{(k,q)\in B} \left\| (I - T)^{-1} \right\|_{L^4 \to L^4} < \infty$$

Moreover, the map

$$q \mapsto (I - T(q,k))^{-1}$$

is Lipschitz from $H^{1,1}(\mathbb{R}^2)$ to $\mathcal{B}(L^4)$.

Proof.

First, the map $L^2(\mathbb{R}^2) \times \mathbb{C} \ni (q,k) \mapsto (I - T(q,k))^{-1} \in \mathcal{B}(L^4)$ is continuous.

Second, the set B is precompact in $L^2(\mathbb{R}^2) \times \mathbb{C}$, so its image under this map is a bounded set.

Finally, Lipschitz continuity follows from boundedness and

$$R(k,q_1) - R(k,q_2) = R(k,q_1)(T(k,q_1) - T(k,q_2))R(k,q_2)$$

where $R(k,q) = (I - T(k,q))^{-1}$. □

Summary

Recall that

$$m^{\pm}(z,k) = m^1(z,k) \pm e_{-k}(z)\overline{m^2(z,k)}$$

We set $w = m^+ - 1$, $q \in H^{1,1}(\mathbb{R}^2)$, and

$$\begin{cases} w - Tw = \partial_{\overline{z}}^{-1}(e_{-k}q) \\ \qquad Tf := \partial_{\overline{z}}^{-1}(e_{-k}q\overline{f}) \end{cases}$$

We have now shown that these equations (and their analogues for m^-) are uniquely solvable and that the maps

$$H^{1,1}(\mathbb{C}) \ni q \mapsto m^{\pm}(z,k) - 1 \in L^4_z(\mathbb{C})$$

are Lipschitz continuous.

We now have (almost all) the necessary tools to study continuity of the direct scattering map

$$H^{1,1}(\mathbb{C}) \ni q \mapsto \mathbf{s}(k) = -\frac{i}{\pi}\int_{\mathbb{C}} e_k(z)\overline{q(z)}m^1(z,k)\,dz$$

since $m^1(z,k) = \frac{1}{2}(m^+(z,k) + m^-(z,k))$

Mixed L^p Estimates

To study the scattering map

$$H^{1,1}(\mathbb{C}) \ni q \mapsto \mathbf{s}(k) = -\frac{i}{\pi}\int_{\mathbb{C}} e_k(z)\overline{q(z)}m^1(z,k)\,dz$$

we need mixed L^p estimates on $m^1(z,k) - 1$. We need a key estimate from Nachman, Regev, and Tataru's paper.

Lemma *Suppose that $p \in (1,2]$ and $f \in L^p(\mathbb{R}^2)$. The estimate*

$$\left|\left(\partial_{\overline{z}}^{-1} f\right)(x)\right| \lesssim (\mathcal{M}f(x))^{\frac{1}{2}}\left(\mathcal{M}\widehat{f}(0)\right)^{\frac{1}{2}}$$

Here $\mathcal{M}f$ is the Hardy-Littlewood Maximal Function

$$(\mathcal{M}f)(x) = \sup_{x\in B}\frac{1}{m(B)}\int_B |f(y)|\,dy.$$

Recall that $\mathcal{M} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Mixed L^p Estimates

We'll usually use this estimate in the form

$$\left|\partial_{\bar{z}}^{-1}(e_{-k}f)(x)\right| \lesssim (\mathcal{M}f(x))^{\frac{1}{2}}\left(\mathcal{M}\widehat{f}(k)\right)^{\frac{1}{2}}$$

Lemma *Suppose that $q \in H^{1,1}(\mathbb{C})$ and that $(\mathcal{M}\widehat{q})(k)$ is finite. Then*

$$\left\|m^1(\,\cdot\,,k) - 1\right\|_{L^4} + \left\|m^2(\,\cdot\,,k)\right\|_{L^4} \leq C(\|q\|_{H^{1,1}})(\mathcal{M}\widehat{q})(k)^{\frac{1}{2}}$$

Moreover, the maps $q \mapsto m^1 - 1$ and $q \mapsto m^2$ are locally Lipschitz maps from $H^{1,1}(\mathbb{C})$ to $L^4(\mathbb{R}^2_z \times \mathbb{R}^2_k)$.

Proof: It suffices to show that $w = m^{\pm} - 1$ obeys the estimate

$$\|w\|_{L^4} \leq C(\|q\|_{H^{1,1}})(\mathcal{M}\widehat{q}(k))^{\frac{1}{2}}$$

As $w = (I-T)^{-1}(\partial_{\bar{z}}(e_k q))$ we have

$$\begin{aligned}\|w\|_{L^4} &\lesssim \left\|(I-T)^{-1}\right\|_{L^4\to L^4}\left\|\partial_{\bar{z}}^{-1}(e_{-k}q)\right\|_{L^4}\\ &\lesssim C(\|q\|_{H^{1,1}})\,\|q\|_{L^2}^{\frac{1}{2}}\,(\mathcal{M}\widehat{q}(k))^{\frac{1}{2}}\end{aligned}$$

Mixed L^p Estimates

Lemma *Suppose that $q \in H^{1,1}(\mathbb{C})$ and that $(\mathcal{M}\widehat{q})(k)$ is finite. Then*

$$\left\| m^1(\,\cdot\,,k) - 1 \right\|_{L^4} + \left\| m^2(\,\cdot\,,k) \right\|_{L^4} \leq C(\|q\|_{H^{1,1}})(\mathcal{M}\widehat{q})(k)^{\frac{1}{2}}$$

Moreover, the maps $q \mapsto m^1 - 1$ and $q \mapsto m^2$ are locally Lipschitz maps from $H^{1,1}(\mathbb{C})$ to $L^4(\mathbb{R}^2_z \times \mathbb{R}^2_k)$.

Proof (continued): As a consequence of the estimate

$$\|w\|_{L^4} \lesssim C(\|q\|_{H^{1,1}})\, \|q\|_{L^2}^{\frac{1}{2}} \left(\mathcal{M}\widehat{q}(k)\right)^{\frac{1}{2}}$$

we obtain

$$\left\| m^1 - 1 \right\|_{L^4(\mathbb{R}^2_z \times \mathbb{R}^2_k)} + \left\| m^2 \right\|_{L^4(\mathbb{R}^2_z \times \mathbb{R}^2_k)} \lesssim C(\|q\|_{H^{1,1}})\, \|q\|_{L^2}^{\frac{1}{2}}$$

Lipschitz continuity of $(m^1 - 1, m^2)$ can be recovered from Lipschitz continuity of the resolvent $(I - T)^{-1}$ and the solution formula.

Continuity of the Scattering Map

With estimates on the scattering solutions in hand, we'll prove continuity of the scattering map

$$q \mapsto \mathbf{s}(k) = -\frac{i}{\pi}\int_{\mathbb{C}} e_k(z)\overline{q(z)}m^1(z,k)\,dz$$

in the following steps:

(1) $\mathcal{S}$ is locally bounded and Lipschitz continuous from $H^{1,1}(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$

(2) For any $p \in [2,\infty)$, $\mathcal{S}$ is locally bounded and Lipschitz continuous from $H^{1,1}(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$

(3) $\mathcal{S}$ is locally bounded and Lipschitz continuous from $H^{1,1}(\mathbb{R}^2)$ to $H^1(R^2)$

(4) $\mathcal{S}$ is locally bounded and Lipschitz continuous from $H^{1,1}(\mathbb{R}^2)$ to $L^{2,1}(\mathbb{R}^2)$

As we'll see, each step provided information needed for the next step. Remember that it suffices to study

$$I(k) = -\frac{i}{\pi}\int_{\mathbb{C}} e_k(z)\overline{q(z)}(m^1(z,k)-1)\,dz$$

$\mathcal{S} : H^{1,1}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ (1 of 3)

We need to show that

$$I(k) = -\frac{i}{\pi} \int e_k(z)\overline{q(z)}(m^1(z,k) - 1)\, dz$$

defines an L^2 function of k, locally Lipschitz as a function of q. From

$$m^1(z,k) - 1 = \partial_{\overline{z}}^{-1}(q(\,\cdot\,)m^2(\,\cdot\,,k))(z)$$

we integrate by parts to get

$$I(k) = -\frac{i}{\pi} \int \left[\partial_{\overline{z}}^{-1}(e_k\overline{q})(z)\right] q(z)m^2(z,k)\, dz$$

From the estimate

$$\left|\partial_{\overline{z}}^{-1}(e_{-k}f)(x)\right| \lesssim (\mathcal{M}f(x))^{\frac{1}{2}} \left(\mathcal{M}\widehat{f}(k)\right)^{\frac{1}{2}}$$

we get

$$|I(k)| \lesssim C(\|q\|_{H^{1,1}}) \left(\mathcal{M}\widehat{\overline{q}}(k)\right)^{\frac{1}{2}} \int (\mathcal{M}\overline{q}(z))^{\frac{1}{2}}\, |q(z)||m^2(z,k)|\, dz$$

$\mathcal{S} : H^{1,1}(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \quad (2 \text{ of } 3)$

Recall

$$I(k) = -\frac{i}{\pi} \int e_k(z) \overline{q(z)} (m^1(z,k) - 1)\, dz$$

We estimate

$$\begin{aligned} |I(k)| &\lesssim C(\|q\|_{H^{1,1}}) \left(\mathcal{M}\widehat{\overline{q}}(k) \right)^{\frac{1}{2}} \int (\mathcal{M}\overline{q}(z))^{\frac{1}{2}} |q(z)| |m^2(z,k)|\, dz \\ &\lesssim C(\|q\|_{H^{1,1}}) \left(\mathcal{M}\widehat{\overline{q}}(k) \right)^{\frac{1}{2}} \|q\|_{L^2}^{\frac{3}{2}} \left\| m^2(\,\cdot\,,k) \right\|_{L^4} \end{aligned}$$

Using the estimate

$$\left\| m^2 \right\|_{L^4(\mathbb{R}^2_z \times \mathbb{R}^2_k)} \lesssim C(\|q\|_{H^{1,1}}) \|q\|_{L^2}^{\frac{1}{2}}$$

we conclude that

$$\|I\|_{L^2} \lesssim C(\|q\|_{H^{1,1}}) \|q\|_{L^2}^2 .$$

$\mathcal{S} : H^{1,1}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ (3 of 3)

To prove Lipschitz continuity, recall

$$I(k) = -\frac{i}{\pi} \int e_k(z)\overline{q(z)}(m^1(z,k) - 1)\, dz.$$

Hence

$$\begin{aligned} I(k;q_1) &- I(k;q_2) \\ &= \frac{i}{\pi} \int \left[q_1(z)\partial_{\overline{z}}^{-1}(e_k\overline{q_1})(z) - q_2(z)\partial_{\overline{z}}^{-1}(e_k\overline{q_2})(z) \right] m^2(z,k;q_1)\, dz \\ &\quad + \frac{i}{\pi} \int \left[q_2(z)\partial_{\overline{z}}^{-1}(e_k\overline{q_2}) \right] \left(m^2(z,k;q^1) - m^2(z,k;q_2) \right) dz \end{aligned}$$

One can show

- $q \to q\partial_{\overline{z}}^{-1}(e_k\overline{q})$ is Lipschitz continuous from $L^2(\mathbb{R}^2)$ to $L^4(\mathbb{R}^2_k, L^{\frac{4}{3}}(\mathbb{R}^2_z))$
- $q \to m^2(z,k;q)$ is Lipschitz continuous from $H^{1,1}(\mathbb{R}^2)$ to $L^4(\mathbb{R}^2_z \times \mathbb{R}^2_k)$

which implies the required Lipschitz continuity.

$\mathcal{S} : H^{1,1}(\mathbb{R}^2) \to L^p(\mathbb{R}^2),\ p > 2 \quad (1 \text{ of } 1)$

Recall

$$I(k) = -\frac{i}{\pi} \int e_k(z)\overline{q(z)}(m^1(z,k) - 1)\, dz.$$

and estimate

$$\begin{aligned} |I(k)| &\lesssim \int \left| \partial_{\overline{z}}^{-1}(e_k q)(z) \right| \left| m^2(z,k) \right| \, dz \\ &\lesssim C(\|q\|_{H^{1,1}}) \int \left[(\mathcal{M}\widehat{\overline{q}})(k)^{\frac{1}{2}} (\mathcal{M}q)(z)^{\frac{1}{2}} \right]^2 dz \\ &\lesssim C(\|q\|_{H^{1,1}}) \, \|q\|_{L^2}^2 \left(\mathcal{M}\widehat{\overline{q}} \right)(k) \end{aligned}$$

Since $\mathcal{M} : L^p \to L^p$ for $p \in (1, \infty)$, the result follows from the Hausdorff-Young inequality.

$\mathcal{S} : H^{1,1}(\mathbb{R}^2) \to H^1(\mathbb{R}^2)$ (1 of 1)

It suffices to estimate $\left\|\partial_{\overline{k}} \mathbf{s}\right\|_{L^2}$ since the *Beurling transform*

$$(\mathbf{S}f)(z) = -\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|z-w|>\varepsilon} \frac{f(w)}{(z-w)^2}\, dw, \quad f \in C_0^\infty(\mathbb{R}^2)$$

extends to a bounded operator on $L^p(\mathbb{R}^2)$, $p \in (1,\infty)$, with

$$\mathbf{S}(\partial_{\overline{z}} f) = \partial_z f$$

We compute, for $q \in \mathcal{S}(\mathbb{R}^2)$,

$$\begin{aligned} \partial_{\overline{k}} \mathbf{s}(k) &= I_1 + I_2 \\ I_1 &= \frac{1}{\pi} \int e_k(z) \overline{z} \overline{q(z)} m^1(z,k)\, dz \qquad I_2 = \frac{i}{\pi} \mathbf{s}(k) \int \overline{q(z) m^2(z,k)}\, dz. \end{aligned}$$

Use the facts that:

- In I_1, $zq(z) \in L^2(\mathbb{R}^2)$
- In I_2, $\mathbf{s} \in L^4(\mathbb{R}^2_k)$ and $q \mapsto \int \overline{q(z) m^2(z,k)}\, dz$ defines a continuous map from $H^{1,1}(\mathbb{R}^2)$ to $L^4(\mathbb{R}^2_k)$.

$\mathcal{S} : H^{1,1}(\mathbb{R}^2) \to L^{2,1}(\mathbb{R}^2)$ (1 of 2)

Using $\partial_{\overline{z}} e_k = i\overline{k} e_k$ and integrating by parts, we see that

$$\overline{k} I(k) = -\frac{1}{\pi} \int e_k(z) \partial_{\overline{z}} \left(\overline{q(z)} (m^1(z,k) - 1) \right) dz = \frac{1}{\pi}(I_1 + I_2)$$

where

$$I_1 = -\int e_k(z) (\partial_{\overline{z}} \overline{q})(z) (m^1(z,k) - 1)\, dz$$

$$I_2 = -\int e_k(z) |q(z)|^2 m^2(z,k)\, dz$$

In I_1, use the facts that $\partial_{\overline{z}} \overline{q} \in L^2(\mathbb{R}^2)$ and $m^1(z,k) - 1 = \partial_{\overline{z}}^{-1}(q(z) m^2(z,k))$.

In I_2, use the equations for m^2 to write

$$I_2 = -\int |q(z)|^2 \partial_z^{-1}(e_k(\,\cdot\,) m^1(\,\cdot\,,k))(z)\, dz = -(I_{21} + I_{22}),$$

$$I_{21} = \int |q(z)|^2 \partial_z^{-1}(e_k q)(z)\, dz$$

$$I_{22} = \int |q(z)|^2 \partial_z^{-1} \left(e_k q(\,\cdot\,)(m^1(\,\cdot\,,k) - 1))(z) \right) dz$$

$\mathcal{S} : H^{1,1}(\mathbb{R}^2) \to L^{2,1}(\mathbb{R}^2)$ (2 of 2)

$$I_{21} = \int |q(z)|^2 \partial_z^{-1}(e_k q)(z)\, dz$$
$$I_{22} = \int |q(z)|^2 \partial_z^{-1}\left(e_k q(\,\cdot\,)(m^1(\,\cdot\,,k) - 1))(z)\right) dz$$

Via "integration by parts" we have

$$I_{21} = -\int e_k(z) q(z) \partial_z^{-1}(|q|^2)(z)\, dz$$

which exhibits I_{21} as the Fourier transform of an L^2 function since $|q|^2 \in L^{\frac{4}{3}}(\mathbb{R}^2)$.

On the other hand,

$$I_{22} = -\int e_k(z) q(z) \left[\partial_z^{-1}(|q(\,\cdot\,)|^2)\right](z)(m^1(z,k) - 1)\, dz$$

Since $q\partial_z^{-1}(|q|^2)$ is an L^2 function, I_{22} is also an L^2 function of k.

$\mathcal{S}^{-1} = \mathcal{S}$ (1 of 2)

We will show that, for $q \in \mathcal{S}(\mathbb{R}^2)$, the solutions (m^1, m^2) of

$$\begin{aligned} \partial_{\overline{z}} m^1(z,k) &= q(z) m^2(z,k) \\ (\partial_z + ik) m^2(z,k) &= \overline{q(z)} m^1(z,k) \\ m^1(z,k) - 1, \quad m^2(z,k) &= \mathcal{O}\left(|z|^{-1}\right) \end{aligned}$$

also solve

$$\begin{aligned} \partial_{\overline{k}} m^1(z,k) &= e_{-k} \mathbf{s}(k) \overline{m^2(z,k)} \\ \partial_{\overline{k}} m^2(z,k) &= e_{-k} \mathbf{s}(k) \overline{m^1(z,k)} \\ m^1(z,k) - 1, \quad m^2(z,k) &= \mathcal{O}\left(|k|^{-1}\right) \end{aligned}$$

where

$$\mathbf{s}(k) = -\frac{i}{\pi} \int_{\mathbb{R}^2} e_k(z) \overline{q(z)} m^1(z,k) \, dz.$$

$\mathcal{S} = \mathcal{S}^{-1}$ (2 of 2)

$$\begin{aligned} \partial_{\bar{z}} m^1(z,k) &= q(z) m^2(z,k) \\ (\partial_z + ik) m^2(z,k) &= \overline{q(z)} m^1(z,k) \\ m^1(z,k) - 1, \quad m^2(z,k) &= \mathcal{O}\left(|z|^{-1}\right) \end{aligned}$$

Let $v^1 = \partial_{\bar{k}} m^1$, $v^2 = \partial_{\bar{k}} m^2$. Differentiate the equations with respect to $\bar{k}$ to find

$$\begin{aligned} \partial_{\bar{z}} v^1 &= q v^2 \\ (\partial_z + ik) v^2 &= \bar{q} v^1 \\ v^1 = \mathcal{O}\left(|z|^{-1}\right), \quad v^2 &= e_{-k} \mathbf{s}(k) + \mathcal{O}\left(|z|^{-1}\right). \end{aligned}$$

Set $w^1(z,k) = \partial_{\bar{k}} m^1 - e_k \mathbf{s} \overline{m^2}$, $w^2 = \partial_{\bar{k}} m^2 - e_{-k} \mathbf{s} \overline{m^1}$ so

$$\begin{aligned} \partial_{\bar{z}} w^1 &= q w^2 \\ (\partial_z + ik) w^2 &= \bar{q} w^1 \end{aligned}$$

where $w_1, w_2 = \mathcal{O}\left(|z|^{-1}\right)$, so $w^1 = w^2 = 0$.

Summary

We have now sketched the proof that $\mathcal{S} : H^{1,1}(\mathbb{R}^2) \to H^{1,1}(\mathbb{R}^2)$ with local Lipschitz continuity.

We have shown that $\mathcal{S} = \mathcal{S}^{-1}$ on $\mathcal{S}(\mathbb{R}^2)$, which we can extend by density to $H^{1,1}(\mathbb{R}^2)$.

This now proves that

$$q_0 \mapsto \mathcal{S}^{-1}\left(e^{2it((\,\cdot\,)^2+(\overline{\cdot})^2)}\mathcal{S}q_0\right)$$

is a continuous map from $H^{1,1}(\mathbb{R}^2)$ to $C(\mathbb{R}, H^{1,1}(\mathbb{R}^2))$ with Lipschitz continuity in q_0.