MA 114 Worksheet #24: Review for Exam 03

1. Find the volume of the following solids.

   (a) The solid obtained by rotating the region bounded by $y = x^2$ and $x = y^2$ about the x-axis,

   Solution: Washer method, $\displaystyle\int_0^1 \pi\left(x - x^4\right)\, dx = \frac{3\pi}{10}$

   (b) The solid obtained by rotating the region bounded by $x = y^2$ and $x = 1$ about the line $x = 1$,

   Solution: Disc method, $\displaystyle\int_0^1 \pi(1 - y^2)^2\, dy = \frac{8\pi}{15}$

   (c) The solid obtained by rotating the region bounded by $y = 4x - x^2$ and $y = 3$ about the line $x = 1$,

   Solution: Washer method, $\displaystyle\int_1^3 2\pi(x-1)\left[x(4-x) - 3\right]\, dx = \frac{8\pi}{3}$

   (d) The solid with circular base of radius 1 and cross-sections perpendicular to the base that are equilateral triangles.

   Solution: The area of an equilateral triangle with side length s is $\sqrt{3}s^2/4$. For a given x, the side length s is $s = 2\sqrt{1-x^2}$. So the volume of this solid is

   $$V = \int_{-1}^1 \sqrt{3}(1 - x^2)\, dx = \frac{4\sqrt{3}}{3}$$

2. Find the area of the surface of revolution obtained by rotating the given curve about the given axis.

   (a) $y = \sqrt{x+1},\ 0 \le x \le 3$; about x-axis,

   Solution: $ds = \sqrt{1 + \dfrac{1}{4(x+1)^2}}\, dx$ so

   $$A = \int_0^3 2\pi\sqrt{x+1}\sqrt{1 + \frac{1}{4(x+1)^2}}\, dx$$

   This is a difficult integral and I wouldn't worry about evaluating it.

(b) $x = 3t^2,\ y = 2t^3,\ 0 \le t \le 5$; about y-axis.

Solution: $ds = 6t\sqrt{t^2+1}$ and

$$A = \int_0^5 2\pi x\, ds = 2\pi \int_0^5 (3t^2) \cdot 6t\sqrt{t^2+1}\, dt = 2\pi \cdot 11616$$

Again, I wouldn't bother actually computing the integral - you will not find something this ugly on the final.

3. Compute the arc length of the following curves.

(a) $x = a\cos^3\theta,\ y = a\sin^3\theta,\ 0 \le \theta \le 2\pi,$

Solution: Since $dx/d\theta = 3a\cos^2\theta\sin\theta$, $dy/d\theta = 3a\sin^2\theta\cos\theta$ we compute

$$ds = 3a\left|\sin\theta\cos\theta\right|\, d\theta$$

so that

$$L = 3a\int_0^{2\pi} ds = 6a$$

(b) $y = \sqrt{2-x^2},\ 0 \le x \le 1$

Solution: This curve is part of a circle of radius $\sqrt{2}$. The arc from $(0,2)$ to $(1,1)$ subtends an angle of $\pi/4$ so the arc length is $\sqrt{2}\pi/4$.

4. Find the centroid of the region bounded by $y = \sqrt{x}$ and $y = x$.

Solution: See the formulas on page 564 of the text.

$$A = \int_0^1 \left(\sqrt{x} - x\right)\, dx = \frac{1}{6}$$

$$\overline{x} = \frac{1}{A}\int_0^1 x(\sqrt{x} - x)\, dx = \frac{2}{5}$$

$$\overline{y} = \frac{1}{A}\int_0^1 \frac{1}{2}(x - x^2)\, dx = \frac{1}{2}$$

5. Find the average value of the function $y = 3\sin(x) + \cos(2x)$ on the interval $[0,\pi]$.

Solution:

$$\frac{1}{\pi}\int_0^{\pi} (3\sin x + \cos 2x)\ dx = \frac{6}{\pi}$$

6. Compute the slope of the tangent line to the curve in Problem 3(a) above, with $a = 8$, at the point $(1, 3\sqrt{3})$. Use this to determine an equation for the tangent line.

Solution: Note the correction to the original WS 24 problem here. The parametric equations are $x = 8\cos^3\theta$, $y = 8\sin^3\theta$, and the point $(1, 3\sqrt{3})$ corresponds to $\theta = \pi/3$. At $\theta = \pi/3$, $dx/d\theta = -24\cos^2\theta\sin\theta = -3\sqrt{3}$ and $dy/d\theta = 24\sin^2\theta\cos\theta = 9$ so the slope of the tangent line is $-\sqrt{3}$. Thus we get

$$y - 3\sqrt{3} = (-\sqrt{3})(x-1).$$

7. Consider the curve given by the parametric equations $(x(t), y(t)) = (t^2, 2t+1)$.

   (a) Find the tangent line to the curve at $(4, -3)$. Put your answer in the form $y = mx + b$.

   Solution: First, $dy/dx = (dy/dt)/(dx/dt) = 2/(2t) = 1/t$ and the point $(4, -3)$ corresponds to $t = -2$. Hence $y + 3 = (-1/2)(x-4)$ or $y = (-1/2)x - 1$.

   (b) Find second derivative $\dfrac{d^2y}{dx^2}$ at $(x, y) = (4, -3)$. Is the curve concave up or concave down near this point?

   Solution: Recall that

   $$\frac{d^2y}{dx^2} = \frac{1}{(dx/dt)}\frac{d}{dt}\left(\frac{dy}{dx}\right)$$

   so in our case

   $$\frac{d^2y}{dx^2} = \frac{1}{2t}\frac{d}{dt}\left(\frac{1}{t}\right) = \frac{-1}{2t^3}.$$

   Evaluating at $t = -2$ we get $d^2y/dx^2 = 1/16 > 0$ so the curve is concave up.