THE CONFORMAL GEOMETRY OF THE SCATTERING OPERATOR

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ABSTRACT. These are notes for a lecture delivered at the Workshop on Conformal Invariants – Geometric and Analytic Aspects, June 10-14, at the National Center for Theoretical Science, Hsinchu, Taiwan. The work on scattering for pseudoconvex domains described here is joint work with Peter Hislop and Siu-Hung Tang. This work is motivated by Graham and Zworski's work which identified certain poles of the scattering operator on asymptotically hyperbolic manifolds as conformally invariant differential operators. We will consider two settings: scattering theory for asymptotically hyperbolic manifolds, as described in the lectures of Colin Guillarmou, and scattering theory for pseudoconvex domains. In each setting we will discuss the geometric information carried by the poles of the scattering operator.

CONTENTS

1. INTRODUCTION

If (X, g) is a complete, non-compact Riemannian manifold with "simple geometry at infinity," the spectral theory of the Laplace operator Δ_g can be studied using techniques of scattering theory. We would like to discuss two examples of this geometric scattering theory of particular relevance to conformal geometry: scattering theory for asymptotically hyperbolic manifolds, such as real hyperbolic space or its quotients by a geometrically finite discrete group, and scattering theory for pseudoconvex domains, such as the complex unit ball. The point of view taken here–in which scattering theory is viewed as a study of boundary value problems for degenerate elliptic operators on a compact manifold with boundary–was advocated by Richard Melrose in his monograph *Geometric Scattering Theory* [11] and has been remarkable fruitful in the study of the Laplacian on complete manifolds.

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2. Asymptotically Hyperbolic Manifolds

In the first case, X is the interior of an $(n+1)$-dimensional manifold with boundary, $\overline{X}$, and the metric g takes the form $x^{-2}h$ where x is a defining function for $\partial\overline{X}$ and h is a smooth, nondegenerate metric on $\overline{X}$. Such a metric is called *conformally compact*. Since any two defining functions differ by a strictly positive $\mathcal{C}^\infty$ function on $\overline{X}$, the conformal class of $h|_M$ is fixed by the Riemannian structure on X. If the sectional curvatures of g approach -1 asymptotically as $p \to M$, then the metric is called *asymptotically hyperbolic*. For an asymptotically hyperbolic metric, one can find coordinates (x, y), where x is a boundary defining function and the coordinates y are constant along integral curves of $\nabla_g x$, so that the metric takes the form

$$g = \frac{dx^2}{x^2} + \frac{H}{x^2} \tag{2.1}$$

where $H = H(x, y, dy)$ has a Taylor series to all orders in x at $x = 0$ (see Proposition 2.1 of [8]; the paper [8] and the thesis [6] also analyzes the scattering operator for asymptotically hyperbolic manifolds in some detail, using the methods of [10]).

Let us describe the spectral theory of the Laplacian on (X, g) when g is asymptotically hyperbolic. We will denote by $\mathcal{C}^\infty(X)$ the smooth functions on X (with no restrictions on behavior as $x \downarrow 0$), by $\dot{\mathcal{C}}^\infty(X)$ the $\mathcal{C}^\infty(X)$ functions which vanish to all orders at $x = 0$, and by $\mathcal{C}^\infty(\overline{X})$ the set of $\mathcal{C}^\infty(X)$ functions with Taylor expansions to all orders at $x = 0$. Choosing coordinates (x, y) as described above, the Laplacian takes the form

$$\Delta_g = -(x\partial_x)^2 + n(x\partial_x) + x^2\Delta_h + xQ(x, y, x\partial_y) \tag{2.2}$$

where $h = H|_M$. and Q is a second-degree polynomial in the differential operators with coefficients smooth in x and y. The operator

$$I(\Delta_g) = -(x\partial_x)^2 + n(x\partial_x) \tag{2.3}$$

is called the *indicial operator* for Δ_g.Note that

$$I(\Delta_g)x^\sigma = \sigma(n - \sigma)x^\sigma \tag{2.4}$$

The operator Δ_g has at most finitely many $L^2(X)$-eigenvalues in the interval $(0, n^2/4)$ and absolutely continuous spectrum in $(n^2/4, \infty)$. Thus the L^2-resolvent $(\Delta_g - z)^{-1}$ is a meromorphic function in the cut plane $\mathbb{C}\backslash[n^2/4, \infty)$. It is convenient to re-parameterize by writing

$$R(s) = (\Delta_g - s(n - s))^{-1}$$

for $\mathrm{Re}(s) > n/2$. Mazzeo and Melrose [10] showed that the resolvent $R(s)$, viewed as a map from $\dot{\mathcal{C}}^\infty(X)$ to $\mathcal{C}^\infty(X)$, admits a meromorphic continuation to $s \in \mathbb{C}$. It follows from their results that, for $\mathrm{Re}(s) > n/2$,

$$R(s) : \dot{\mathcal{C}}^\infty(X) \to x^s\mathcal{C}^\infty(\overline{X}) \tag{2.5}$$

a fact which we will use later.

Consider the eigenvalue problem

$$(\Delta_g - s(n - s))\, u = 0$$

In analogy to the theory of expansions about a regular singular point for ordinary differential equations, it is not unreasonable to expect that this problem has solutions $u \in \mathcal{C}^\infty(X)$ having the form

$$u = x^{n-s}F + x^s G$$

where F and G belong to $\mathcal{C}^\infty(\overline{X})$, provided that $n-2s$ is not an integer (for such "exceptional" points we expect logarithmic terms to arise; this observation will play a very important role later). For $\mathrm{Re}(s) = n/2$ such solutions are generalized eigenfunctions of the Laplace operator and play a key role in scattering theory. Let

$$f = F|_{\partial\overline{X}}$$
$$g = G|_{\partial\overline{X}}$$

For such solutions, assuming only that $n-2s$ is not an integer, the complete Taylor expansions of F and G are determined by f and g. This is easily seen using the form (2.2) of the Laplacian and the identity In fact, a great deal more is true. Let us denote by $\mathcal{E}$ the 'exceptional' set of points where either (1) $2s-n$ not an integer or (2) $s(n-s)$ not an eigenvalue of the Laplacian. If $s \notin \mathcal{E}$, the "Dirichlet problem"

$$(\Delta_g - s(n-s))\, u = 0 \tag{2.6}$$

$$u = x^{n-s}F + x^s G \tag{2.7}$$

$$F|_{\partial\overline{X}} = f \tag{2.8}$$

for a given function $f \in \mathcal{C}^\infty(\partial\overline{X})$ has a *unique* solution. The mapping

$$\mathcal{P}(s) : \mathcal{C}^\infty(\partial\overline{X}) \ni f \to u \in \mathcal{C}^\infty(X)$$

is called the *Poisson operator*. This implies that the mapping

$$S(s) : \mathcal{C}^\infty(\partial\overline{X}) \to \mathcal{C}^\infty(\partial\overline{X})$$

given by

$$S(s)f = G|_{\partial\overline{X}} \tag{2.9}$$

is well-defined. This mapping is called the *absolute scattering operator*. It follows immediately from its definition that

$$S(s)S(n-s) = I$$

and it follows from the boundary pairing formula (2.11) that $S(s)$ is unitary.

Note that the scattering operator depends in a rather explict way on the choice of defining function: if $\widetilde{x} = e^\psi x$ for $\psi \in \mathcal{C}^\infty(\overline{X})$ and $\Upsilon = \psi|_{\partial\overline{X}}$ then

$$\widetilde{S}(s) = e^{-s\Upsilon} S(s) e^{(n-s)\Upsilon}. \tag{2.10}$$

Thus, $S(s)$ is naturally viewed as a map from conformal densities of weight $(n-s)$ to conformal densities of weight s.

For $\mathrm{Re}(s) = n/2$ the uniqueness is a consequence of the *boundary pairing formula*. Suppose that v_1 and v_2 are functions in $\mathcal{C}^\infty(X)$ of the form

$$v_i = x^{n-s}F_i + x^s G_i$$

where F_i and G_i belong to $\mathcal{C}^\infty(\overline{X})$. Suppose further that the v_i are "almost" generalized eigenfunctions in the sense that

$$(\Delta_g - s(n-s))\, v_i = r_i \in \dot{\mathcal{C}}^\infty(X),\ i = 1,2$$

A simple computation with Green's formula for the Laplacian shows that

$$\int_X \left(v_1\overline{r_2} - v_2\overline{r_1}\right)\, dg = (2s-n)\int_{\partial\overline{X}} \left(f_1\overline{f_2} - g_1\overline{g_2}\right)\, dh \tag{2.11}$$

(use the observation that $\overline{v_i} = x^{n-s}\overline{F_i} + x^s\overline{G_i}$ if $\mathrm{Re}(s) = n/2!$). Thus, if u_1 and u_2 are both solutions of the Dirichlet problem (2.6)-(2.8), the function $v = u_1 - u_2$ takes the form (2.7) with $f = 0$. It follows that

$$v = x^s G$$

and from the boundary pairing formula with $u_1 = u_2 = v$ we get $f = 0$, hence $v \in L^2(X, g)$, hence $v = 0$.

A generalization of the boundary pairing formula can be used to show that the Dirichlet problem (2.6)-(2.8) also has a unique solution when $\mathrm{Re}(s) > n/2$ and $s \notin \mathcal{E}$. At the exceptional points $s \in \mathcal{E}$, it is expected that the scattering operator will have poles. It can also be shown that $S(s)$ is self-adjoint for s real.

Graham and Zworski (see the lecture [4] and paper [5], and see also the beautiful thesis of Guillarmou [6] and subsequent paper [7]) analyzed the behavior of the scattering operator near the points $s \in \mathcal{E}$ with $\mathrm{Re}(s) > n/2$ and $2s - n \in \mathbb{N}$. In [5], Graham and Zworski proved two remarkable theorems Denote by P_k the kth conformally invariant power of the Laplacian on the conformal manifold $(M, [h])$. Recall that if $n = \dim M$ is even, Branson's Q-curvature [1] is a quantity constructed from the Riemann curvature tensor and its covariant derivatives that generalizes the scalar curature in two dimensions and obeys the transformation law.

$$e^{n\Upsilon}\widehat{Q} = Q + P_{n/2}\Upsilon \tag{2.12}$$

The first result is

Theorem 2.1. *Let (X, g) be an asymptotically hyperbolic manifold of dimension $n + 1$ with conformal infinity $(M, [h])$, and let $S(s)$ be the scattering operator for the Laplacian Δ_g. Let $s = n/2 + k$, suppose that $(n/2)^2 - k^2$ is not an eigenvalue of the Laplacian, and suppose that $k \leq n/2$ if n is even. The scattering operator $S(s)$ has an infinite-rank pole with residue*

$$\operatorname*{Res}_{s=n/2+k} S(s) = c_k P_k$$

The self-adjointness of the operators P_k is an easy consequence.

Theorem 2.2. *Suppose that n is even. Then*

$$c_{n/2} Q = S(n)1$$

From this formula, Graham and Zworski give another proof of the transformation law (2.12) and show that $\int_M Q\, dh$ is a conformal invariant. Subsequently, Graham and Fefferman [3] were able to deduce many of the same results by formal power series arguments that did not make use of scattering theory.

Graham and Zworski's analysis is based on a careful consideration of the Poisson operator $\mathcal{P}(s)$. In particular, they show that the Poisson operator is analytic in s so long as $s(n - s)$ is not an eigenvalue, and that if $s = n/2 + l/2$ (corresponding to the crossing of indicial roots)

$$\mathcal{P}(s)f = x^{n/2-l/2}F + Gx^{n/2+l/2}\log x$$

They further show that the residue of the scattering operator at $s = n/2 + l/2$, denoted p_l can be calculated from the formula

$$G|_{\partial\overline{X}} = 2p_l$$

The Poisson operator can be constructed as follows. Given $f \in \mathcal{C}^\infty(\partial \overline{X})$, we can construct a smooth function u_1 on X having asymptotic series

$$u_1 \sim \sum_{j \geq 0} x^{n-s+j} f_j(y)$$

so that

$$r_1 = (\Delta_g - s(n-s))\, u_1 \in \dot{\mathcal{C}}^\infty(X)$$

The functions f_j can be computed iteratively using the form (2.5) of Δ_g. On the other hand, using the mapping property (2.5), the function

$$u_2 = -R(s) u_1$$

(here we must assume that $s(n-s)$ is not an eigenvalue of Δ_g) has an asymptotic series of the form

$$u_2 \sim \sum_{j \geq 0} x^{s+j} g_j(y)$$

so $u = u_1 + u_2$ solves the Dirichlet problem (2.6)-(2.8). The identification of p_l as the residue of the scattering operator, and the calculation of the principal symbol, follow from a formal power series analysis.

3. Strictly Pseudoconvex Domains

Next, we consider scattering theory for the Laplacian on a pseudoconvex domain. As we will see there are some remarkable analogies with the case of asymptotically hyperbolic manifolds but also some important differences. Epstein, Melrose, and Mendoza [2] constructed the resolvent family $R(s)$ for a large class of manifolds including the Laplacian on a pseudoconvex domain. Subsequently, Melrose [12] announced further results on the scattering operator and its meromorphic continuation. Here we will identify certain poles of the scattering operator on a strictly pseudoconvex domain as conformally invariant differential operators whose principal symbol is a power of the sub-Laplacian on its boundary. The work described here is joint with Peter Hislop and Siu-Hung Tang.

Recall that a strictly pseudoconvex domain Ω in $\mathbb{C}^n$ is an open subset of $\mathbb{C}^n$ with the following property: there is a smooth function φ with $\varphi < 0$ in Ω strictly, $\varphi^{-1}(0) = \partial\Omega$, and the Hermitian matrix

$$h_{jk} = \frac{\partial^2 \varphi}{\partial z_j \partial z_{\overline{k}}}$$

is strictly positive definite in $\overline{\Omega}$. If $g = -\log(-\varphi)$ then the Hermitian metric

$$ds^2 = g_{j\overline{k}}\; dz^j dz^{\overline{k}}$$

defines a complete Kähler metric on Ω. Clearly

$$g_{j\overline{k}} = \frac{\varphi_{j\overline{k}}}{-\varphi} + \frac{\varphi_j \varphi_{\overline{k}}}{\varphi^2}$$

The positive Laplacian is the operator

$$\Delta_g = -g^{j\overline{k}} \partial_j \partial_{\overline{k}}$$

while the Kähler form is given by

$$\omega = \frac{i}{2} \left(\frac{\partial \overline{\partial} \varphi}{-\varphi} + \frac{\partial \varphi \wedge \overline{\partial} \varphi}{\varphi^2} \right).$$

Let $\iota : M \to \overline{\Omega}$ be the natural inclusion. The manifold M carries a CR-structure given by the form

$$\theta = \iota^*(-i\partial\varphi)$$

By pseudoconvexity, the Levi form

$$d\theta = \iota^*(-i\partial\overline{\partial}\varphi)$$

is nondegenerate on $H = \ker\theta$, the holomorphic tangent bundle for M. The form

$$\psi = \theta \wedge (d\theta)^{n-1}. \tag{3.1}$$

is nondegenerate and defines a volume element for M.

Changing the defining function φ for $\partial\Omega$ preserves the conformal class of the CR-structure on M. If $\overline{\varphi}$ is another defining function for $\partial\Omega$, we have $\overline{\varphi} = e^{\psi}\varphi$ for a smooth function ψ. It is not difficult to see that $\overline{\theta} = e^{\Upsilon}\theta$ and $d\overline{\theta} = e^{\Upsilon}d\theta$ where $\Upsilon = \psi|_M$.

Lee and Melrose [9] showed that, in boundary normal coordinates (where $x = -\varphi$ and coordinates y are constant along integral curves of $\nabla_g\varphi$) the Laplacian associated to the Kahler metric takes the form

$$\Delta_g = I(x\partial_x) + x\left(-r(x\partial_x)^2 + \Box_b + V\right) + x^2 R_2(x, y, x\partial_x, \partial_y) \tag{3.2}$$

where

$$I(x\partial_x) = -(x\partial_x)^2 + n(x\partial_x) \tag{3.3}$$

is the indicial operator, r is a smooth function,

$$\Box_b = \overline{\partial}_b^*\overline{\partial}_b \tag{3.4}$$

is the boundary Laplacian associated to the Levi form, and

$$V = \frac{1}{2}i(n-1)T$$

where T is the unique vector field on M with $\theta(T) = 1$ and $T \lrcorner\, d\theta = 0$. There is a formal similarity with the problem for asymptotically hyperbolic manifolds with the important differences that (1) the underlying algebra of differential operators is generated by the vector fields $x\partial_x$ and ∂_y rather than $x\partial_x$ and $x\partial_y$ and (2) the boundary operators that occur are degenerate elliptic.

The Laplacian Δ_g has continuous spectrum in $[n^2/4, \infty)$. Epstein, Melrose, and Mazzeo showed that the resolvent operator (for a large class of problems including the one considered here!)

$$R(s) = (\Delta_g - s(n-s))^{-1}$$

admits a meromorphic continuation to the complex s-plane and that the resolvent has the mapping property

$$R(s) : \dot{\mathcal{C}}^\infty(\Omega) \to x^s\mathcal{C}^\infty(\Omega_{1/2})$$

where the space $\mathcal{C}^\infty(\Omega_{1/2})$ consists of those functions in $\mathcal{C}^\infty(X)$ having Taylor expansions to all orders in $x^{1/2}$ (see Proposition 12.10 in [2]). Using the form (3.2) of the Laplacian, we can refine this mapping property to assert that

$$R(s) : \dot{\mathcal{C}}^\infty(\Omega) \to x^s\mathcal{C}^\infty(\overline{\Omega}).$$

Moreover, using Green's formula

$$\int_U (u_1\Delta u_2 - u_2\Delta u_1)\ \omega^n = \int_{\partial U} (u_1(\nu u_2) - u_2(\nu u_1))\ \nu \lrcorner\, \omega^n$$

where ν is the inward unit normal, we can prove a boundary pairing formula essentially identical to (2.11). We obtain the following uniqueness result.

Proposition 3.1. *Suppose that* $\mathrm{Re}(s) = n/2$, *and* $s \neq n/2$. *Then the Dirichlet problem*

$$(\Delta_g - s(n-s))\, u = 0 \tag{3.5}$$

$$u = x^{n-s}F + x^s G \tag{3.6}$$

$$F|_{\partial\overline{\Omega}} = f \tag{3.7}$$

where F *and* G *belong to* $\mathcal{C}^\infty(\partial\overline{\Omega})$, *has a unique solution.*

This uniqueness result allows us to define the scattering operator

$$S(s) : \mathcal{C}^\infty(\partial\Omega) \to \mathcal{C}^\infty(\partial\Omega)$$

for $\mathrm{Re}(s) = n/2$, $s \neq n/2$ by

$$S(s)f = G|_{\partial\overline{\Omega}} \tag{3.8}$$

and the Poisson operator

$$\mathcal{P}(s) : \mathcal{C}^\infty(\partial\Omega) \to \mathcal{C}^\infty(\Omega)$$

for the same s by

$$\mathcal{P}(s)f = u.$$

These two maps enjoy many of the same formal properties as their counterparts on asymptotically hyperbolic manifolds. In particular, the scattering operator is conformally covariant in the sense that if $\widetilde{x} = e^\psi x$ for $\psi \in \mathcal{C}^\infty(\overline{\Omega})$ and $\Upsilon = \psi|_{\partial\overline{\Omega}}$ then

$$\widetilde{S}(s) = e^{-s\Upsilon} S(s) e^{(n-s)\Upsilon}. \tag{3.9}$$

Moreover, $S(s)$ is unitary for $s = n/2$ and can be extended to $\mathrm{Re}(s) > n/2$ so long as $s(n-s)$ is not an eigenvalue of Δ_g and $2s - n \notin \mathbb{N}$.

The same formal power series arguments use to construct the Poisson operator for $\mathrm{Re}(s) > n/2$ in the asymptotically hyperbolic case carry over to the case of the Laplacian on a pseudoconvex domain. The Poisson operator in this range is also studied (again, for a large class of problems that includes our particular case) in §15 of [2], although these authors do not construct $\mathcal{P}(s)$ for the 'exceptional' points which play a key role here.

By following closely the formal analysis in [5] and using the form (3.2), we have shown:

Theorem 3.2. *Let* $S(s)$ *be the scattering operator for the Laplacian on a strictly pseudoconvex domain* Ω *in* $\mathbb{C}^n$. *The scattering operator* $S(s)$ *has residues at* $s = n/2 + k$ *of the form*

$$\operatorname*{Res}_{s=n/2+k} S(s) = c_k P_k$$

where P_k *is a differential operator with principal symbol* $\sigma(P_k) = \sigma(\Box_b^{2k})$.

From the transformation law (3.9) it follows that the operators P_k are conformally covariant. That they are self-adjoint with respect to the volume form $\theta \wedge (d\theta)^{n-1}$ follows from the corresponding self-adjointness statement for the scattering operator.

Thus, in the special case of pseudoconvex domains, the connection between scattering theory and (conformal) CR-geometry of the boundary persists. In future work we hope to obtain further information about the conformally invariant operators and extend our results to complete complex manifolds having a natural "CR-conformal infinity."

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