RELATIVE PERTURBATION THEORY FOR DIAGONALLY DOMINANT MATRICES

MEGAN DAILEY*, FROILÁN M. DOPICO†, AND QIANG YE‡

Abstract. In this paper, strong relative perturbation bounds are developed for a number of linear algebra problems involving diagonally dominant matrices. The key point is to parameterize diagonally dominant matrices via their off-diagonal entries and diagonally dominant parts and to consider small relative componentwise perturbations of these parameters. This allows us to obtain new relative perturbation bounds for the inverse, the solution to linear systems, the symmetric indefinite eigenvalue problem, the singular value problem, and the nonsymmetric eigenvalue problem. These bounds are much stronger than traditional perturbation results, since they are independent of either the standard condition number or the magnitude of eigenvalues/singular values. Together with previously derived perturbation bounds for the LDU factorization and the symmetric positive definite eigenvalue problem, this paper presents a complete and detailed account of relative structured perturbation theory for diagonally dominant matrices.

Key words. accurate computations, diagonally dominant matrices, diagonally dominant parts, inverses, linear systems, eigenvalues, singular values, relative perturbation theory

AMS subject classifications. 65F05, 65F15, 65F35, 15A09, 15A12, 15A18

1. Introduction.

Diagonally dominant matrices form an important class of matrices that arise in a large number of applications. Finite difference discretizations of elliptic differential operators, Markov chains, and graph Laplacians are some typical examples of this type of matrices, among many others. Indeed, diagonal dominance is often equivalent to some natural physical property of a practical problem. Diagonally dominant matrices have some nice numerical and theoretical properties, as explained in [17, 20, 21, 22]. For instance, a strictly diagonally dominant matrix is nonsingular and its LU factorization always exists, which can be stably computed without carrying out any pivoting. Furthermore, inverses, and hence condition numbers, of diagonally dominant matrices can be bounded in terms of the minimal diagonal dominance [29, 35, 37].

Recent works have shown that diagonally dominant matrices may enjoy much better numerical properties than those presented in classical texts [17, 20]. The novel idea of these works is to exploit the structure of diagonally dominant matrices via their parametrization in terms of the off-diagonal entries and the diagonally dominant parts [1, 38]. With the new parametrization, stronger perturbation bounds and more accurate algorithms have been obtained for certain linear algebra problems in [7, 11, 38, 39]. Specifically, a relative perturbation theory is presented in [39] for the eigenvalues of a symmetric positive semi-definite diagonally dominant matrix (i.e., a symmetric diagonally dominant matrix with nonnegative diagonals), which simply bounds the relative variation of the eigenvalues by the relative perturbation of the matrix parameters, without involving any condition number, constant, or amplifying

*Indiana University Kokomo, 2300 S. Washington St., Kokomo, IN 46904-9003, USA. E-mail: medailey@iuk.edu. Research supported in part by NSF Grant DMS-1318633.

†Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM and Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain. E-mail: dopico@math.uc3m.es. Research supported in part by *Ministerio de Economía y Competitividad* of Spain under grant MTM2012-32542.

‡Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA. E-mail: qiang.ye@uky.edu. Research supported in part by NSF Grant DMS-1318633.

factor. In [11], a structured perturbation theory is presented for the LDU factorization of diagonally dominant matrices that provides simple and strong bounds on the entrywise relative variations for the diagonal matrix D and the normwise relative variations for the factors L and U. This result has been recently improved in an essential way in [7] by allowing the use of a certain pivoting strategy which guarantees that the factors L and U are always well-conditioned. Computationally, a new algorithm is presented in [38] that accurately computes the LDU factorization of diagonally dominant matrices with entrywise accurate factors D and normwise accurate factors L and U. This algorithm can be combined with the algorithms presented in [8] to compute the singular values with relative errors in the order of machine precision. In fact, the algorithm for the LDU factorization in [38] can be combined also with the algorithms in [4, 12, 13] to compute with high relative accuracy solutions to linear systems and solutions to least squares problems involving diagonally dominant matrices, and eigenvalues of symmetric diagonally dominant matrices.

In this paper, we shall significantly broaden this study by establishing strong relative perturbation bounds for a number of other linear algebra problems involving diagonally dominant matrices. First, a perturbation bound is presented for the inverse of a diagonally dominant matrix that is independent of any condition number. As a direct consequence, we shall also establish a perturbation bound for the solution to the linear system $Ax = b$, which is governed by a certain condition number that is of order one for most vectors b and is always smaller than the traditional condition number of A. Then, the relative eigenvalue perturbation bound for a symmetric positive semi-definite diagonally dominant matrix in [39] is generalized to the indefinite case. Next, relative perturbation bounds for the singular values of any diagonally dominant matrix are obtained. These bounds are equal to a dimensional constant times the perturbation of the parameters, without involving any other amplifying factor. Lastly, we shall derive relative perturbation bounds for the eigenvalues of a nonsymmetric diagonally dominant matrix, which are still dependent on the Wilkinson eigenvalue condition number as usual [9, 17], but independent of the magnitude of the eigenvalue itself. A remarkable feature of all the bounds presented in this paper is that they are finite rigorous bounds, i.e., they are not asymptotic bounds valid only for infinitesimal perturbations.

We shall rely heavily on the LDU perturbation results from [7, 11]. Indeed, most of the new bounds in this paper are derived starting from the perturbation bounds for the LDU factorization. In addition, some other results included in [14] will also play a relevant role. Our methods can be directly adapted to the structured perturbation problem where a general matrix, i.e., not necessarily diagonally dominant, is perturbed in such a way that a rank revealing decomposition XDY of this matrix [8] is changed with small entrywise relative variations for the diagonal matrix D and small normwise relative variations for the factors X and Y (see also [13]). We do not insist in this approach, but, as an example, we present one such result in Theorem 6.3. Clearly, other strong perturbation bounds can also be derived for matrices under such a structured perturbation.

This paper can be seen as a contribution to one of the most fruitful lines of research in matrix perturbation theory in the last two decades: the derivation of perturbation bounds much stronger than the traditional ones when structure-preserving perturbations of relevant types of structured matrices are considered (see, for instance, [1, 2, 3, 6, 11, 18, 19, 23, 24, 27, 28, 34, 39] and the references therein). Even more, we can say that this manuscript belongs to a more specific class of recent research works in structured matrix perturbation theory: those that represent certain structured matrices by a proper set of parameters (different than the matrix entries), in such a way that tiny perturbations of these parameters produce tiny variations of some interesting quantities in linear algebra. Apart from the references on diagonally dominant matrices mentioned above, other references dealing with *perturbations via parameters* are [10, 16, 30, 31, 32] for eigenvalues and eigenvectors of tridiagonal matrices parameterized by their bidiagonal LDU factorizations, and [25] for eigenvalues and singular values of totally nonnegative matrices parameterized by their bidiagonal decompositions.

The rest of the paper is organized as follows. In Section 2, an overview of diagonally dominant matrices and related perturbation results for their LDU factorizations from [7, 11] are presented. In addition, Section 2 includes a numerical example that illustrates why the parametrization via off-diagonal entries and diagonally dominant parts is essential to get strong perturbation bounds. We develop relative perturbation bounds for the inverse and solutions to linear systems in Section 3, for the symmetric indefinite eigenvalue problem in Section 4, for the singular value problem in Section 5, and for the nonsymmetric eigenvalue problem in Section 6. Finally, we conclude by presenting some remarks in Section 7.

Next, we present the notation used in this paper.

NOTATION: We consider only real matrices and we denote by $\mathbb{R}^{m\times n}$ the set of $m \times n$ real matrices. The entries of a matrix A are a_{ij} or A_{ij}, and $|A|$ is the matrix with entries $|a_{ij}|$. The inequality $A \geq B$ for matrices means $a_{ij} \geq b_{ij}$ for all i, j, and the inequality $v \geq w$ for vectors means $v_i \geq w_i$ for all the entries of the vectors. Analogously, the inequality $v \geq 0$ for the vector v means $v_i \geq 0$ for all its entries. We use MATLAB notation for submatrices. That is, $A(i : j, k : l)$ denotes the submatrix of A formed by rows i through j and columns k through l. We also use $A(i', j')$ to denote the submatrix of A formed by deleting row i and column j from A. Let $\alpha = [i_1, i_2, \ldots, i_p]$, where $1 \leq i_1 < i_2 < \cdots < i_p \leq m$, and $\beta = [j_1, j_2, \ldots, j_q]$, where $1 \leq j_1 < j_2 < \cdots < j_q \leq n$. Then $A(\alpha, \beta)$ denotes the submatrix of A that consists of rows $i_1, i_2, \ldots, i_p$ and columns $j_1, j_2, \ldots, j_q$. We denote by I_s the $s \times s$ identity matrix, by 0_s the $s \times s$ zero matrix, and by $0_{p\times q}$ the $p \times q$ zero matrix. We will simply write I and 0 when their sizes are clear from the context. Five matrix norms will be used: $\|A\|_{\max} = \max_{ij} |a_{ij}|$, $\|A\|_1 = \max_j \sum_i |a_{ij}|$, $\|A\|_\infty = \max_i \sum_j |a_{ij}|$, $\|A\|_F = (\sum_{i,j} |a_{ij}|^2)^{1/2}$, and the spectral norm $\|A\|_2$. The condition numbers of a nonsingular matrix A in any of these norms are denoted as $\kappa_i(A) := \|A\|_i \, \|A^{-1}\|_i$, for $i = \max, 1, \infty, F, 2$. The sign of $x \in \mathbb{R}$ is $\operatorname{sign}(x)$, where $\operatorname{sign}(0)$ is defined to be 1.

2. Preliminaries and example. In this section, we give an overview of diagonally dominant matrices and present some results proved recently in [7, 11] that will be used in the subsequent sections. More information on diagonally dominant matrices can be found in [7, Section 2] and [11, Section 2], and the references therein. In addition, at the end of this section, we present and discuss an example which illustrates why the use of a proper parametrization is essential to obtain strong perturbation bounds for diagonally dominant matrices. We first define diagonally dominant matrices.

DEFINITION 2.1. *A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is said to be row diagonally dominant if $|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|$ for $i = 1, \ldots, n$, and is said to be column diagonally dominant if $|a_{ii}| \geq \sum_{j\neq i} |a_{ji}|$ for $i = 1, \ldots, n$.*

For brevity, we will consider only row diagonally dominant matrices, although the results we present hold for column diagonally dominant matrices with obvious modifications or by taking transposes.

An idea that has played an important role in deriving strong perturbation bounds for diagonally dominant matrices is to reparameterize the matrix in terms of its diagonally dominant parts and off diagonal entries (see [38]).

DEFINITION 2.2. (1) *Given a matrix* $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ *and a vector* $v = [v_i] \in \mathbb{R}^n$, *we use* $\mathcal{D}(M, v)$ *to denote the matrix* $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ *whose off-diagonal entries are the same as* M *(i.e.,* $a_{ij} = m_{ij}$ *for* $i \neq j$*) and whose ith diagonal entry is* $a_{ii} = v_i + \sum_{j\neq i} |m_{ij}|$ *for* $i = 1, \ldots, n$.

(2) *Given a matrix* $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, *we denote by* $A_D \in \mathbb{R}^{n\times n}$ *the matrix whose off-diagonal entries are the same as* A *and whose diagonal entries are zero. Then, letting* $v_i = a_{ii} - \sum_{j\neq i} |a_{ij}|$, *for* $i = 1, \ldots, n$, *and* $v = [v_1, v_2, \ldots, v_n]^T \in \mathbb{R}^n$, *we have*

$$A = \mathcal{D}(A_D, v)$$

and we call it the representation of A *by its diagonally dominant parts* v *and off-diagonal entries* A_D.

Clearly, $v \geq 0$ if and only if A is row diagonally dominant and its diagonal entries are nonnegative. We will use very often the condition $v \geq 0$ as assumption without referring explicitly to its meaning. For most problems (i.e., the LDU factorization, inverses, linear systems, and the singular value problem), by considering an equivalent problem for SA with $S = \mathrm{diag}\{\mathrm{sign}(a_{ii})\}$, we can restrict ourselves to diagonally dominant matrices A with nonnegative diagonal without loss of generality. For the eigenvalue problem, however, we need to consider in general diagonally dominant matrices with diagonal entries of any sign. To properly parameterize such matrices, we need the signs of the diagonal entries (i.e., S) as well; we shall leave the details of this more general case to Section 4 when we study the symmetric indefinite eigenvalue problem.

Several of our results are based on the perturbation bounds for the LDU factorization recently obtained in [7, 11]. We first recall that if the LU, or LDU, factorization of a nonsingular matrix exists, then it is unique. However, for singular matrices, when an LU, or LDU, factorization exists, it is not unique in general. In this case, in order to study its perturbation properties, we need to consider the following unique form of the LDU factorization (see [11, Definition 1]).

DEFINITION 2.3. *A matrix* $A \in \mathbb{R}^{n\times n}$ *with rank* r *is said to have LDU factorization if there exist a unit lower triangular matrix* $L_{11} \in \mathbb{R}^{r\times r}$, *a unit upper triangular matrix* $U_{11} \in \mathbb{R}^{r\times r}$, *and a nonsingular diagonal matrix* $D_{11} \in \mathbb{R}^{r\times r}$ *such that* $A = LDU$, *where*

$$L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & I_{n-r} \end{bmatrix}, \qquad D = \begin{bmatrix} D_{11} & 0 \\ 0 & 0_{n-r} \end{bmatrix}, \qquad U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & I_{n-r} \end{bmatrix}.$$

It is easy to see that if this form of LDU factorization exists, then it is unique. For a row diagonally dominant matrix $A \in \mathbb{R}^{n\times n}$, applying any diagonal pivoting strategy (i.e., pivoting with simultaneous and equal row and column permutations) will result in PAP^T that has a unique LDU factorization in the sense of Definition 2.3, where P is the permutation matrix defined by the pivoting strategy. For the purposes of this work, we consider a pivoting strategy suggested in [33] for matrices with other structures, and used for first time in [38] for general row diagonally dominant matrices. This strategy is called *column diagonal dominance pivoting*. Let $A^{(1)} = A$ and let $A^{(k)} = [a_{ij}^{(k)}] \in \mathbb{R}^{n\times n}$ denote the matrix obtained after $(k-1)$ stages of Gaussian elimination have been performed on A, i.e., all entries below the diagonal in the first $k-1$ columns of $A^{(k)}$ are zero. It is well known ([11, Theorem 1] or [20]) that the Schur complement $A^{(k)}(k:n, k:n)$ is row diagonally dominant. Then, there is at least one column of this Schur complement that is column diagonally dominant, i.e., $|a_{ii}^{(k)}| - \sum_{j=k, j\neq i}^{n} |a_{ji}^{(k)}| \geq 0$ for some $i \geq k$. So, the column diagonal dominance pivoting scheme permutes into the pivot position (k,k) the maximal diagonal entry that is column diagonally dominant. That is, at step k, after the permutation, we have

$$\left|a_{kk}^{(k)}\right| = \max_{k \leq i \leq n} \left\{ \left|a_{ii}^{(k)}\right| : |a_{ii}^{(k)}| - \sum_{j=k, j\neq i}^{n} |a_{ji}^{(k)}| \geq 0 \right\},$$

where we still use $A^{(k)} = [a_{ij}^{(k)}]$ to denote the matrix after the permutation. With this pivoting strategy, at the end, we obtain a row diagonally dominant factor U as usual, but now L is column diagonally dominant. Hence, by [33], L, U and their inverses can be bounded as

$$\|L\|_{\max} = 1,\ \|L\|_1 \leq 2,\ \|L\|_\infty \leq n,\ \|L^{-1}\|_{\max} = 1,\ \|L^{-1}\|_1 \leq n,\ \|L^{-1}\|_\infty \leq n, \tag{2.1}$$
$$\|U\|_{\max} = 1,\ \|U\|_1 \leq n,\ \|U\|_\infty \leq 2,\ \|U^{-1}\|_{\max} = 1,\ \|U^{-1}\|_1 \leq n,\ \|U^{-1}\|_\infty \leq n\,. \tag{2.2}$$

The bounds for the inverses in (2.1) and (2.2) follow from Proposition 2.1 in [33], which states that the inverses of triangular diagonally dominant (either by rows or columns) matrices with ones on the diagonal have the absolute values of their entries bounded by one. It is worth observing that this result follows immediately from a classic and more general result to be found in [22, Theorem 2.5.12], where a proof is given for strictly diagonally dominant matrices but can be easily extended to any nonsingular diagonally dominant matrix.

The bounds in (2.1,2.2) imply that the LDU factorization of a row diagonally dominant matrix A obtained by column diagonal dominance pivoting is always a rank-revealing decomposition [8], which is of fundamental interest for performing accurate computations. For all the linear algebra problems we consider here, since the permuted matrix PAP^T coming from any diagonal pivoting strategy results in trivially equivalent problems, we can assume that the row diagonally dominant matrix we consider is arranged for column diagonal dominance pivoting, i.e., A has the permutation P applied already. More importantly, the unique LDU factorization obtained under this pivoting scheme is stable under componentwise perturbations of the diagonally dominant parts and off-diagonal entries. Indeed, the following perturbation bounds are obtained in [7, 11].

THEOREM 2.4. [7, Theorem 3.2]-[11, Theorem 3] *Let* $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ *be such that* $v \geq 0$. *Suppose that* A *has* LDU *factorization* $A = LDU$, *where* $L = [l_{ij}]$, $D = \mathrm{diag}(d_1, \ldots, d_n)$, *and* $U = [u_{ij}]$. *Let* $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ *be a matrix that satisfies*

$$|\tilde{v} - v| \leq \epsilon v \quad \textit{and} \quad |\tilde{A}_D - A_D| \leq \epsilon |A_D|, \tag{2.3}$$

for some positive ϵ *with* $(12n+1)\epsilon < 1$. *Then, we have*

1. $\tilde{A}$ *is row diagonally dominant with nonnegative diagonal entries,* $\mathrm{rank}(\tilde{A}) = \mathrm{rank}(A)$, *and* $\tilde{A}$ *has* LDU *factorization* $\tilde{A} = \tilde{L}\tilde{D}\tilde{U}$, *where* $\tilde{L} = [\tilde{l}_{ij}]$, $\tilde{D} = \mathrm{diag}(\tilde{d}_1, \ldots, \tilde{d}_n)$, *and* $\tilde{U} = [\tilde{u}_{ij}]$;

2. For $i = 1, \ldots, n$, $\tilde{d}_i = d_i(1 + w_i)$ with

$$\left(\frac{1-\epsilon}{1+\epsilon}\right)^n - 1 \le w_i \le \left(\frac{1+\epsilon}{1-\epsilon}\right)^n - 1, \tag{2.4}$$

and, in particular, $|\tilde{d}_i - d_i| \le \dfrac{2n\epsilon}{1-2n\epsilon}|d_i|$, *for* $i = 1, \ldots, n$;

3. $|\tilde{u}_{ij} - u_{ij}| \le 3n\epsilon$, *for* $1 \le i, j \le n$, *and* $\dfrac{\|\tilde{U} - U\|_\infty}{\|U\|_\infty} \le 3n^2\epsilon$;

4. and, if A is arranged for column diagonal dominance pivoting, then

$$\|\tilde{L} - L\|_1 \le \frac{n(8n-2)\epsilon}{1-(12n+1)\epsilon} \quad \textit{and} \quad \frac{\|\tilde{L} - L\|_1}{\|L\|_1} \le \frac{n(8n-2)\epsilon}{1-(12n+1)\epsilon}.$$

The main remark on the relative bounds presented in Theorem 2.4 is that they do not depend on any condition number, neither of the matrix A nor of its factors, and so, they imply that for *any* row diagonally dominant matrix, small componentwise perturbations as in (2.3) always produce small relative changes in the LDU factors. Observe also that $v \ge 0$ and the fact that $0 \le \epsilon < 1$ in (2.3) imply immediately that $\tilde{v} \ge 0$, which is the reason why the perturbations in (2.3) preserve the diagonally dominant structure and the nonnegativity of the diagonal entries.

We shall also use in the rest of the paper the following lemma, which combines Lemmas 3, 4, and 7 of [11], and studies the perturbation of the determinant and certain minors of diagonally dominant matrices with nonnegative diagonals under structured perturbations of type (2.3).

LEMMA 2.5. *Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ be such that $v \ge 0$ and let $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ satisfy* (2.3) *for some ϵ with $0 \le \epsilon < 1$. Then:*

(a) $\det \tilde{A} = (\det A)(1 + \eta_1)\cdots(1 + \eta_n)$, *where* $|\eta_j| \le \epsilon$ *for* $1 \le j \le n$;

(b) *If $\alpha = [i_1, i_2, \ldots, i_t]$, where $1 \le i_1 < i_2 < \cdots < i_t \le n$, then*

$$\det \tilde{A}(\alpha,\alpha) = (\det A(\alpha,\alpha))(1+\beta_1)\cdots(1+\beta_t), \quad \textit{where } |\beta_j| \le \epsilon, \quad \textit{for } 1 \le j \le t;$$

(c) *If $k + 1 \le p, q \le n$ and $p \ne q$, then*

$$\begin{aligned}&\left|\det \tilde{A}([1:k,p],[1:k,q]) - \det A([1:k,p],[1:k,q])\right| \\ &\qquad\qquad \le 2\left((1+\epsilon)^{k+1} - 1\right)\det A([1:k,p],[1:k,p]).\end{aligned}$$

To finish this section, we present an example to illustrate why perturbations via parameters of type (2.3) may be expected to lead to stronger bounds than general perturbations or perturbations that only preserve the diagonally dominant property. For brevity, Example 2.6 focuses only on singular values, but similar examples can be devised for the other linear algebra problems considered in this paper.

EXAMPLE 2.6. Let us consider the following row diagonally dominant matrix A, whose vector of diagonally dominant parts is denoted by v_A:

$$A = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.002 & 1 \\ 2 & 0.5 & 2.5 \end{bmatrix}, \qquad v_A = \begin{bmatrix} 0 \\ 0.002 \\ 0 \end{bmatrix}.$$

The *following two row diagonally dominant matrices*

$$B = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.001 & 1 \\ 2 & 0.5 & 2.5 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3.0015 & -1.5015 & 1.5 \\ -1 & 2.002002 & 1 \\ 2 & 0.5 & 2.5 \end{bmatrix}$$

are very close to A in an standard entrywise sense, since they satisfy

$$|A-B| \leq 5 \cdot 10^{-4}\,|A| \quad \text{and} \quad |A-C| \leq 10^{-3} \cdot |A|.$$

However, their vectors of diagonally dominant parts are

$$v_B = \begin{bmatrix} 0 & 0.001 & 0 \end{bmatrix}^T \quad \text{and} \quad v_C = \begin{bmatrix} 0 & 0.002002 & 0 \end{bmatrix}^T,$$

and we see that v_B is very different from v_A, while v_C is very close to v_A. More precisely,

$$|v_A - v_B| = 0.5\, v_A \quad \text{and} \quad |v_A - v_C| = 10^{-3}\, v_A. \tag{2.5}$$

Thus, in plain words, we can say that C is a nearby perturbation of A in the sense of diagonally dominant parts and off-diagonal entries, while B is not. That is, B is a nearby perturbation of A only in the traditional entrywise sense.

In the following table, we show the singular values, $\sigma_1 \geq \sigma_2 \geq \sigma_3$, of A, B, and C in the usual decreasing order (all digits shown in this table are exact):

	A	B	C
σ_1	4.641	4.640	4.642
σ_2	2.910	2.909	2.910
σ_3	$6.663 \cdot 10^{-4}$	$3.332 \cdot 10^{-4}$	$6.673 \cdot 10^{-4}$

The first key remark on this table is that the smallest singular values of A and B do *not* agree in a single digit, despite the fact that A and B are very close to each other and both are row diagonally dominant. Therefore, just preserving the diagonally dominant property may not be enough to get good perturbation properties. In contrast, the smallest singular values of A and C do agree in two digits. More precisely, for $i = 1, 2, 3$,

$$\max_i \frac{|\sigma_i(A) - \sigma_i(B)|}{\sigma_i(A)} = 0.49989 \quad \text{and} \quad \max_i \frac{|\sigma_i(A) - \sigma_i(C)|}{\sigma_i(A)} = 1.4444 \cdot 10^{-3}.$$

The behavior we observe in this example is not by chance, since in Section 5 we will show that for row diagonally dominant matrices, tiny relative perturbations of diagonally dominant parts and off-diagonal entries always result in tiny relative variations of the singular values, independently of their magnitudes.

3. Bounds for inverses and solutions to linear systems. The perturbation theory for the inverse of a matrix A and for the solution to linear systems $Ax = b$ is well established and can be found in many books on numerical linear algebra [9, 17, 20, 36]. The classical perturbation bounds of a general matrix depend on the traditional condition number $\kappa(A) := \|A\|\,\|A^{-1}\|$ for normwise perturbations, while for entrywise perturbations, they are governed by the Bauer-Skeel condition number $\|\,|A^{-1}|\,|A|\|$. Both of these condition numbers may be unbounded in general. By focusing on row diagonally dominant matrices and parameterized entrywise perturbations of their diagonally dominant parts and off-diagonal entries, we shall prove in this section new entrywise perturbation bounds on the inverse that are *independent of any condition number*. Similarly, for the solution to linear systems, this structured perturbation allows us to present normwise bounds that are dependent on a smaller condition number that is almost always a moderate number of order one.

The main idea in this section is simple: noting that the entries of A^{-1} can be expressed in terms of minors of A [21], we utilize the perturbation results for determinants presented in Lemma 2.5 to obtain the following entrywise perturbation bounds for the inverse of a row diagonally dominant matrix.

THEOREM 3.1. *Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ be such that $v \geq 0$ and suppose that A is nonsingular. Let $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ be such that*

$$|\tilde{v} - v| \leq \epsilon v \quad \textit{and} \quad |\tilde{A}_D - A_D| \leq \epsilon |A_D|, \quad \textit{for some} \quad 0 \leq \epsilon < 1.$$

Then $\tilde{A}$ is nonsingular and if $2n\epsilon < 1$, we have for $1 \leq i, j \leq n$,

$$\left|(\tilde{A}^{-1})_{ij} - (A^{-1})_{ij}\right| \leq \frac{(3n-2)\epsilon}{1-2n\epsilon} \left|(A^{-1})_{jj}\right|. \tag{3.1}$$

Proof. By Lemma 2.5(a), we have $\det \tilde{A} = (\det A)(1+\eta_1)\cdots(1+\eta_n) \neq 0$, since $|\eta_j| \leq \epsilon < 1$ for all j. Therefore, $\tilde{A}$ is nonsingular.

To prove (3.1), we consider $j \neq i$ first. Without loss of generality, we assume that $i = n-1$ and $j = n$, since this can always be obtained via proper simultaneous *row and column* permutations of A^{-1}, corresponding, respectively, to the same simultaneous *row and column* permutations in A. Using Lemma 2.5(c) with $k = n-2$, $p = n-1$, and $q = n$, we have

$$|\det \tilde{A}(j', i') - \det A(j', i')| \leq 2\left((1+\epsilon)^{n-1} - 1\right) \det A(j', j'). \tag{3.2}$$

It follows from this and Lemma 2.5(a)-(b) that

$$\begin{aligned}
(\tilde{A}^{-1})_{ij} - (A^{-1})_{ij} &= \frac{(-1)^{i+j} \det \tilde{A}(j', i')}{\det \tilde{A}} - \frac{(-1)^{i+j} \det A(j', i')}{\det A} \\
&= \frac{(-1)^{i+j} \det \tilde{A}(j', i')}{(\det A)(1+\eta_1)\cdots(1+\eta_n)} - \frac{(-1)^{i+j} \det A(j', i')}{\det A} \\
&= \frac{(-1)^{i+j} \chi(\det \tilde{A}(j', i') - \det A(j', i'))}{\det A} + (\chi - 1)\frac{(-1)^{i+j} \det A(j', i')}{\det A},
\end{aligned}$$

where $\chi := \dfrac{1}{(1+\eta_1)\cdots(1+\eta_n)}$. Noting that $|\chi - 1| \leq \dfrac{1}{(1-\epsilon)^n} - 1$ and using (3.2), we have

$$\begin{aligned}
\left|(\tilde{A}^{-1})_{ij} - (A^{-1})_{ij}\right| &\leq \frac{|\chi| |\det \tilde{A}(j', i') - \det A(j', i')|}{|\det A|} + |\chi - 1| \frac{|\det A(j', i')|}{|\det A|} \\
&\leq \frac{2\left((1+\epsilon)^{n-1} - 1\right) |\chi| |\det A(j', j')|}{|\det A|} + |\chi - 1| \frac{|\det A(j', i')|}{|\det A|} \\
&= 2\left((1+\epsilon)^{n-1} - 1\right) |\chi| \left|(A^{-1})_{jj}\right| + |\chi - 1| \left|(A^{-1})_{ij}\right| \\
&\leq \frac{2\left((1+\epsilon)^{n-1} - 1\right)}{(1-\epsilon)^n} \left|(A^{-1})_{jj}\right| + \left[\frac{1}{(1-\epsilon)^n} - 1\right] \left|(A^{-1})_{ij}\right| \\
&\leq \frac{2(n-1)\epsilon}{1-2n\epsilon} \left|(A^{-1})_{jj}\right| + \frac{n\epsilon}{1-n\epsilon} \left|(A^{-1})_{ij}\right|,
\end{aligned}$$

where we have used (see [20, Chapter 3]) that

$$\frac{(1+\epsilon)^{n-1} - 1}{(1-\epsilon)^n} \leq \frac{(n-1)\epsilon/(1-(n-1)\epsilon)}{1 - n\epsilon/(1-n\epsilon)} \leq \frac{(n-1)\epsilon}{1-2n\epsilon}.$$

From [11, Theorem 1(e)], we have $|(A^{-1})_{ij}| \leq |(A^{-1})_{jj}|$, which leads to (3.1) for $i \neq j$.

Finally, we prove (3.1) for $i = j$. Again, we use Lemma 2.5(a)-(b) to prove that

$$(\tilde{A}^{-1})_{ii} = \frac{\det \tilde{A}(i', i')}{\det \tilde{A}} = \frac{(\det A(i', i'))(1+\beta_1)\cdots(1+\beta_{n-1})}{(\det A)(1+\eta_1)\cdots(1+\eta_n)}$$
$$= (A^{-1})_{ii} \frac{(1+\beta_1)\cdots(1+\beta_{n-1})}{(1+\eta_1)\cdots(1+\eta_n)},$$

where $|\eta_j| \leq \epsilon < 1$ and $|\beta_j| \leq \epsilon < 1$. According to [20, Lemma 3.1], this equality can be written as

$$(\tilde{A}^{-1})_{ii} = (A^{-1})_{ii}\,(1+\theta_{2n-1}), \quad \text{where} \quad |\theta_{2n-1}| \leq \frac{(2n-1)\epsilon}{1-(2n-1)\epsilon}\,.$$

Therefore, $\left|(\tilde{A}^{-1})_{ii} - (A^{-1})_{ii}\right| = |\theta_{2n-1}|\,\left|(A^{-1})_{ii}\right|$, and

$$\left|(\tilde{A}^{-1})_{ii} - (A^{-1})_{ii}\right| \leq \frac{(2n-1)\epsilon}{1-(2n-1)\epsilon}\left|(A^{-1})_{ii}\right| \leq \frac{(3n-2)\epsilon}{1-2n\epsilon}\left|(A^{-1})_{ii}\right|,$$

which completes the proof. □

We note that the assumption $2n\epsilon < 1$ in Theorem 3.1 is not essential and is only made to simplify the bound. Note also that Theorem 3.1 gives that small relative perturbations in the data $\mathcal{D}(A_D, v)$ result in small relative perturbations in the diagonal entries of the inverse. However, the perturbation of an off-diagonal entry can only be guaranteed to be small relative to the diagonal entry in the corresponding column of the inverse, rather than relative to the off-diagonal entry itself. This might seem unsatisfactory at a first glance, but again the diagonally dominant structure allows us to prove in Corollary 3.2 that the bound (3.1) leads to very satisfactory relative normwise bounds for the inverse, which are completely independent of any condition number.

Corollary 3.2. *Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ be such that $v \geq 0$ and suppose that A is nonsingular. Let $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ be such that*

$$|\tilde{v} - v| \leq \epsilon v \quad \text{and} \quad |\tilde{A}_D - A_D| \leq \epsilon |A_D|, \qquad \text{for some } 0 \leq \epsilon < 1/(2n).$$

Let $\|\cdot\|$ be either the 1-norm, the ∞-norm, or the Frobenius norm. Then

$$\frac{\|\tilde{A}^{-1} - A^{-1}\|}{\|A^{-1}\|} \leq \frac{n(3n-2)\epsilon}{1-2n\epsilon}\,.$$

Proof. Theorem 1(e) in [11] implies $|(A^{-1})_{ij}| \leq |(A^{-1})_{jj}|$ for all i, j. Thus, it follows from Theorem 3.1 that for $1 \leq i, j \leq n$

$$\left|(\tilde{A}^{-1})_{ij} - (A^{-1})_{ij}\right| \leq \frac{(3n-2)\epsilon}{1-2n\epsilon} \max_{k,l} |(A^{-1})_{k,l}|.$$

Then,

$$\|\tilde{A}^{-1} - A^{-1}\| \leq n\frac{(3n-2)\epsilon}{1-2n\epsilon} \max_{k,l} |(A^{-1})_{kl}| \leq n\frac{(3n-2)\epsilon}{1-2n\epsilon}\|A^{-1}\|\,.$$

□

With the results of Corollary 3.2, we can now present perturbation bounds for the solution to linear systems, whose coefficient matrices are row diagonally dominant.

THEOREM 3.3. *Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ be such that $v \geq 0$ and suppose that A is nonsingular. Let $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ be such that*

$$|\tilde{v} - v| \leq \epsilon v \quad \text{and} \quad |\tilde{A}_D - A_D| \leq \epsilon |A_D|, \qquad \text{for some } 0 \leq \epsilon < 1/(2n). \tag{3.3}$$

Let $\|\cdot\|$ be either the 1-norm or the ∞-norm. Let $b, \delta b \in \mathbb{R}^{n\times 1}$ be vectors such that $\|\delta b\| \leq \epsilon \|b\|$ and consider the following two linear systems

$$Ax = b \quad \text{and} \quad \tilde{A}\tilde{x} = b + \delta b.$$

Then,

$$\frac{\|\tilde{x} - x\|}{\|x\|} \leq \left[\frac{(3n^2 - 2n + 1)\epsilon + (3n^2 - 4n)\epsilon^2}{1 - 2n\epsilon}\right] \frac{\|A^{-1}\|\,\|b\|}{\|x\|}. \tag{3.4}$$

Proof. Observe that $\tilde{x} - x = (\tilde{A}^{-1} - A^{-1})b + \tilde{A}^{-1}\delta b$. Then, applying Corollary 3.2 gives,

$$\begin{aligned}
\|\tilde{x} - x\| &\leq \|\tilde{A}^{-1} - A^{-1}\|\,\|b\| + \|\tilde{A}^{-1}\|\,\|\delta b\| \\
&\leq \|\tilde{A}^{-1} - A^{-1}\|\,\|b\| + \left[\|\tilde{A}^{-1} - A^{-1}\| + \|A^{-1}\|\right]\epsilon\|b\| \\
&\leq \frac{n(3n-2)\epsilon}{1 - 2n\epsilon}\|A^{-1}\|\,\|b\| + \left[\frac{n(3n-2)\epsilon}{1 - 2n\epsilon}\|A^{-1}\| + \|A^{-1}\|\right]\epsilon\|b\| \\
&\leq \left[\frac{n(3n-2)\epsilon}{1 - 2n\epsilon} + \epsilon\left(\frac{n(3n-2)\epsilon}{1 - 2n\epsilon} + 1\right)\right]\|A^{-1}\|\,\|b\|.
\end{aligned}$$

Simplifying, this bound leads to (3.4). □

Theorem 3.3 shows that the sensitivity of the linear system $Ax = b$ to parameterized perturbations of type (3.3) is mainly determined by $\|A^{-1}\|\,\|b\|/\|x\|$. For general unstructured matrices, the condition number $\kappa(A, b) := \|A^{-1}\|\,\|b\|/\|x\|$ measures the normwise sensitivity of the solution x when only b is perturbed and A remains unchanged. It is immediate to see that $\kappa(A, b) \leq \kappa(A)$ always holds, but much more important is to note that if $\kappa(A) \gg 1$, then $\kappa(A, b) \ll \kappa(A)$ for most vectors b, that is, the condition number $\kappa(A, b)$ is usually a moderate number compared to $\kappa(A)$. This fact is well-known in numerical linear algebra and it was noted for first time in [5]. Some additional discussions on this point can be found in [13, Section 3.2].

4. Bounds for eigenvalues of symmetric matrices. In this section, we present perturbation bounds for eigenvalues of symmetric diagonally dominant matrices under parameterized perturbations of type (2.3). A first point to keep in mind is that if a matrix A enjoys, simultaneously, the properties of symmetry and row diagonal dominance, then A must be both *row and column* diagonally dominant. These properties give us two additional properties which are essential in this section: (1) the LDU decomposition of A inherits the symmetry, i.e., $A = LDL^T$, and; (2) since $L = U^T$, the L factor satisfies the entrywise perturbation bounds in Theorem 2.4(3), instead of only the normwise bounds in Theorem 2.4(4). Note also that, in this case, column diagonal dominance pivoting coincides with complete diagonal pivoting.

A second point to be remarked is that in [39], a strong relative perturbation bound has already been obtained for the eigenvalues of symmetric diagonally dominant matrices with nonnegative diagonals (hence positive semidefinite). More precisely, it is shown in [39] that if a symmetric diagonally dominant matrix $A = \mathcal{D}(A_D, v)$ with $v \geq 0$ and a perturbed symmetric matrix $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v})$ satisfy $|\tilde{v} - v| \leq \epsilon v$ and $|\tilde{A}_D - A_D| \leq \epsilon |A_D|$ for some ϵ with $0 \leq \epsilon < 1$, then the relative perturbation in the eigenvalues is bounded precisely by ϵ. That is, if $\lambda_1 \geq \cdots \geq \lambda_n$ are the eigenvalues of A and $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_n$ are the eigenvalues of $\tilde{A}$, then

$$|\tilde{\lambda}_i - \lambda_i| \leq \epsilon \lambda_i, \quad \text{for } i = 1, \ldots, n. \tag{4.1}$$

This result is certainly strong and simple, but the techniques used in [39] for proving (4.1) rely heavily on the positive semidefinite character of the matrix A and we do not see how to generalize them to cover symmetric indefinite diagonally dominant matrices. In this section, we will use an approach completely different to the one in [39] to show that a relative perturbation bound similar to (4.1) holds for the eigenvalues of symmetric indefinite diagonally dominant matrices.

If A is symmetric indefinite diagonally dominant, then A has both negative and positive diagonal entries and the parametrization introduced in Definition 2.2 is no longer useful. In this case, it is more appropriate to define the diagonally dominant parts using the absolute values of the diagonal entries, i.e., $v_i := |a_{ii}| - \sum_{j \neq i} |a_{ij}|$. However, a_{ii} can not be obtained from a_{ij} $(j \neq i)$ and v_i defined this way. We need to take the signs of the diagonal entries as additional parameters to define the diagonal entries and, hence, the whole matrix. Thus, we generalize Definition 2.2 to include these additional parameters as follows.

Definition 4.1. *For any $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, let A_D be the matrix whose off-diagonal entries are the same as A and whose diagonal entries are zero. Let*

$$v_i = |a_{ii}| - \sum_{j \neq i} |a_{ij}|, \text{ for } i = 1, \ldots, n,$$
$$S = \operatorname{diag}(\operatorname{sign}(a_{11}), \ldots, \operatorname{sign}(a_{nn})).$$

Then, A is uniquely determined from the parameters A_D, $v = [v_i] \in \mathbb{R}^n$, and S, and we write $A = \mathcal{D}(A_D, v, S)$ to indicate that A is given via these parameters.

With this parametrization, note that row diagonal dominance is equivalent to $v \geq 0$. Next, we introduce Lemmas 4.2 and 4.3, which are simple auxiliary results needed in the proof of the main result in this section, i.e., Theorem 4.4.

Lemma 4.2. *Let $y \geq 0$ and $0 \leq \epsilon < 1$ be real numbers. Then,*

$$\left(\frac{1+\epsilon}{1-\epsilon}\right)^y - 1 \geq 1 - \left(\frac{1-\epsilon}{1+\epsilon}\right)^y.$$

Proof. Let $x = \left(\frac{1+\epsilon}{1-\epsilon}\right)^y$ and observe $x > 0$. Thus, $x + \frac{1}{x} \geq 2$ and, hence, $x - 1 \geq 1 - \frac{1}{x}$. □

Lemma 4.3. *Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. If A has LDU factorization $A = LDU$ in the sense of Definition 2.3, then $U = L^T$ and $A = LDL^T$.*

Proof. Let $r = \operatorname{rank}(A)$ and let

$$L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & I_{n-r} \end{bmatrix}, \qquad D = \begin{bmatrix} D_{11} & 0 \\ 0 & 0_{n-r} \end{bmatrix}, \qquad U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & I_{n-r} \end{bmatrix},$$

with $L_{11}, D_{11}, U_{11} \in \mathbb{R}^{r\times r}$, be the LDU factorization of A in the sense of Definition 2.3. Partition A accordingly as

$$A = \left[\begin{array}{cc} A_{11} & A_{21}^T \\ A_{21} & A_{22} \end{array} \right].$$

Then $A_{11} = L_{11}D_{11}U_{11}$ is the unique LDU factorization of the nonsingular matrix A_{11}. Since A_{11} is symmetric, we have $U_{11} = L_{11}^T$. Furthermore, it follows from $A_{21} = L_{21}D_{11}U_{11}$ and $A_{21}^T = L_{11}D_{11}U_{12}$ that $U_{12} = L_{21}^T$. Therefore $U = L^T$ and $A = LDL^T$. $\square$

We now present the main theorem of this section in which we consider a perturbation of $A = \mathcal{D}(A_D, v, S)$ that has small relative errors in each component of A_D, v, and S. Since S is a diagonal matrix of ± 1, this necessarily implies that S is unperturbed, which means that the signs of the diagonal entries of the matrix are preserved under the perturbation.

THEOREM 4.4. *Let $A = \mathcal{D}(A_D, v, S) \in \mathbb{R}^{n\times n}$ be a symmetric matrix such that $v \geq 0$. Let $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}, S)$ be another symmetric matrix that satisfies*

$$|\tilde{v} - v| \leq \epsilon v \quad \textit{and} \quad |\tilde{A}_D - A_D| \leq \epsilon |A_D|, \quad \textit{for some } 0 \leq \epsilon < 1.$$

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_n$ be the eigenvalues of A and $\tilde{A}$, respectively. If $5n^3\epsilon < 1$, then

$$|\tilde{\lambda}_i - \lambda_i| \leq (2\nu + \nu^2)|\lambda_i|, \quad \textit{for } i = 1, \ldots, n, \tag{4.2}$$

where $\nu = \dfrac{4n^3\epsilon}{1 - n\epsilon}$.

Proof. First, if P is the permutation matrix defined by any diagonal pivoting scheme for A that ensures existence of the LDU factorization in the sense of Definition 2.3 (e.g. the complete pivoting), we can consider PAP^T and $P\tilde{A}P^T$, which do not change the eigenvalues nor the perturbation assumptions. Therefore, we can assume without loss of generality that A is already arranged such that its LDU factorization exists. Observe that S is assumed to be unperturbed. Multiplying S on the left with the matrices A and $\tilde{A}$, we get two diagonally dominant matrices with nonnegative diagonals $C = \mathcal{D}(C_D, v_C)$ and $\tilde{C} = \mathcal{D}(\tilde{C}_D, \tilde{v}_C)$, where

$$C = SA, \;\; C_D = SA_D, \;\; v_C = v,$$

$$\tilde{C} = S\tilde{A}, \;\; \tilde{C}_D = S\tilde{A}_D, \;\; \tilde{v}_C = \tilde{v}.$$

Then,

$$|\tilde{C}_D - C_D| \leq \epsilon |C_D| \qquad \text{and} \qquad |\tilde{v}_C - v_C| \leq \epsilon v_C.$$

Since A is symmetric, it has the LDU factorization with structure $A = LDL^T$ by Lemma 4.3. In addition, C has LDU factorization, $C = L_C D_C U_C$, which satisfies $L_C = SLS$, $D_C = SD$, and $U_C = L^T$, because

$$C = SA = SLDL^T = (SLS)(SD)L^T. \tag{4.3}$$

Now, apply Theorem 2.4(1) to C and $\tilde{C}$ to obtain that $\tilde{C}$ has LDU factorization, which is denoted as $\tilde{C} = \tilde{L}_C\tilde{D}_C\tilde{U}_C$. This implies that $\tilde{A}$ also has LDU factorization, which by Lemma 4.3 has the structure $\tilde{A} = \tilde{L}\tilde{D}\tilde{L}^T$. The same argument we used in (4.3) leads to $\tilde{L}_C = S\tilde{L}S$, $\tilde{D}_C = S\tilde{D}$, and $\tilde{U}_C = \tilde{L}^T$. Next, we apply Theorem 2.4(2)-(3) to C and $\tilde{C}$ by taking into account $D_C = SD$, $U_C = L^T$, $\tilde{D}_C = S\tilde{D}$, and $\tilde{U}_C = \tilde{L}^T$, and, with the notation $L = [l_{ij}]$, $D = \mathrm{diag}[d_i]$, $\tilde{L} = [\tilde{l}_{ij}]$, and $\tilde{D} = \mathrm{diag}[\tilde{d}_i]$, we get

$$\tilde{d}_i = d_i(1 + w_i), \quad \text{with } \left(\frac{1-\epsilon}{1+\epsilon}\right)^n - 1 \le w_i \le \left(\frac{1+\epsilon}{1-\epsilon}\right)^n - 1, \tag{4.4}$$

for $i = 1, \ldots, n$, and

$$|\tilde{l}_{ij} - l_{ij}| \le 3n\epsilon, \quad \text{for } 1 \le i, j \le n, \text{ i.e., } \|\tilde{L} - L\|_{\max} \le 3n\epsilon. \tag{4.5}$$

Set $\gamma_i = \sqrt{1 + w_i} - 1$ and observe that $\tilde{d}_i = d_i(1 + \gamma_i)^2$. From (4.4), we get

$$\left(\frac{1-\epsilon}{1+\epsilon}\right)^{n/2} - 1 \le \gamma_i \le \left(\frac{1+\epsilon}{1-\epsilon}\right)^{n/2} - 1$$

and, from Lemma 4.2, we have

$$|\gamma_i| \le \left(\frac{1+\epsilon}{1-\epsilon}\right)^{n/2} - 1 \le \frac{1}{(1-\epsilon)^n} - 1 \le \frac{n\epsilon}{1 - n\epsilon}\,.$$

Now, set $W = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$. Then, we can write $\tilde{D}$ as

$$\tilde{D} = (I + W)D(I + W), \quad \text{with } \|W\|_{\max} \le \frac{n\epsilon}{1 - n\epsilon}. \tag{4.6}$$

Thus, letting $\Delta_L = \tilde{L} - L$, we have

$$\begin{aligned} \tilde{A} = \tilde{L}\tilde{D}\tilde{L}^T &= (L + \Delta_L)(I + W)D(I + W)(L + \Delta_L)^T \\ &= [L + \Delta_L + LW + \Delta_L W]\, D\, [L + \Delta_L + LW + \Delta_L W]^T \\ &= (I + F)LDL^T(I + F)^T \\ &= (I + F)\, A\, (I + F)^T, \end{aligned} \tag{4.7}$$

where $F = \Delta_L L^{-1} + LWL^{-1} + \Delta_L W L^{-1}$. Since L is column diagonally dominant, then $\|L\|_{\max} = 1$ and $\|L^{-1}\|_{\max} = 1$ by (2.1). These bounds, combined with (4.5) and (4.6), yield

$$\begin{aligned} \|F\|_2 \le \|F\|_F &\le n\|F\|_{\max} \\ &\le n^2 \left[\|\Delta_L\|_{\max}\|L^{-1}\|_{\max} + \|LW\|_{\max}\|L^{-1}\|_{\max} + \|\Delta_L\|_{\max}\|WL^{-1}\|_{\max}\right] \\ &\le n^2 \left[3n\epsilon + \frac{n\epsilon}{1 - n\epsilon} + 3n\epsilon\left(\frac{n\epsilon}{1 - n\epsilon}\right)\right] = \frac{4n^3\epsilon}{1 - n\epsilon} =: \nu. \end{aligned}$$

Since $n^3\epsilon < 1/5$, we have $\|F\|_2 < 1$, which implies $I + F$ is nonsingular. Hence, we can apply [14, Theorem 2.1], which states that if $\tilde{A} = (I + F)A(I + F)^T$ for a nonsingular matrix $(I + F)$, then

$$|\tilde{\lambda}_i - \lambda_i| \le |\lambda_i|\, \|(I + F)(I + F)^T - I\|_2, \quad \text{for } i = 1, \ldots, n. \tag{4.8}$$

Note that

$$\|(I + F)(I + F)^T - I\|_2 = \|F + F^T + FF^T\|_2 \le 2\|F\|_2 + \|F\|_2^2 \le 2\nu + \nu^2.$$

The theorem is proved by combining this inequality with (4.8). □

Theorem 4.4 demonstrates that the relative perturbations of the eigenvalues are proportional to ϵ and are independent of any condition number. However, the bound in (4.2) is weaker than the one in (4.1) proved in [39] for positive semidefinite matrices, since (4.2) contains the dimensional factor n^3. This is mostly the result of bounding the 2-norm of various matrices and vectors from the max norm. It is obviously pessimistic but it is not clear how it can be improved with our current approach.

5. Bounds for singular values. We consider in this section perturbation bounds for singular values of nonsymmetric row diagonally dominant matrices with nonnegative diagonals. Classic perturbation bounds for the singular values of a general matrix A are obtained as by-products of the eigenvalue perturbation theory of symmetric matrices [36] just by applying this theory to

$$B = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix} \quad \text{or} \quad A^T A.$$

However, this approach cannot be followed here, owing to the fact that A being row diagonally dominant does not imply that B or $A^T A$ is diagonally dominant. So, we need to develop a different approach and for this purpose we follow a three-step procedure similar to the one used in the proof of Theorem 4.4: in a first step the perturbation of the LDU factorization is considered via Theorem 2.4, in a second step the bounds for the LDU factors are used to express $\tilde{A}$ as a multiplicative perturbation of A (see (4.7)), and the final step employs on this expression the multiplicative perturbation results from [14]. This allows us to prove Theorem 5.1. Note that in Theorem 5.1 the matrix A is not symmetric and, so, A is only row diagonally dominant, instead of being simultaneously row and column diagonally dominant as in Theorem 4.4. This partially explains why the bound presented in Theorem 5.1 is weaker than the one in Theorem 4.4 if the matrix is symmetric.

THEOREM 5.1. *Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ be such that $v \geq 0$ and let $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ be such that*

$$|\tilde{v} - v| \leq \epsilon v \quad \text{and} \quad |\tilde{A}_D - A_D| \leq \epsilon |A_D|, \quad \text{for some } 0 \leq \epsilon < 1.$$

Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ and $\tilde{\sigma}_1 \geq \tilde{\sigma}_2 \geq \cdots \geq \tilde{\sigma}_n$ be the singular values of A and $\tilde{A}$, respectively, and let

$$\nu := \frac{2n^{5/2}(4n-1)}{1-(12n+1)\epsilon}\epsilon.$$

If $0 \leq \nu < 1$, then

$$|\tilde{\sigma}_i - \sigma_i| \leq (2\nu + \nu^2)\,\sigma_i, \quad \text{for } i = 1, \ldots, n.$$

Proof. As in the proof of Theorem 4.4, we can assume without loss of generality that A is arranged for column diagonal dominance pivoting. So, A has LDU factorization and, by Theorem 2.4, $\tilde{A}$ has also LDU factorization. Let $A = LDU$ and $\tilde{A} = \tilde{L}\tilde{D}\tilde{U}$ be these factorizations, and use the notation $D = \operatorname{diag}(d_1, \ldots, d_n)$ and $\tilde{D} = \operatorname{diag}(\tilde{d}_1, \ldots, \tilde{d}_n)$. Then, Theorem 2.4 implies

$$\tilde{d}_i = d_i(1 + w_i), \quad \text{with } |w_i| \leq \frac{2n\epsilon}{1-2n\epsilon} \text{ for } i = 1, \ldots, n, \tag{5.1}$$

$$\|\Delta_U\|_{\max} \le 3n\epsilon, \quad \text{with } \Delta_U := \tilde{U} - U, \tag{5.2}$$

and

$$\|\Delta_L\|_2 \le \sqrt{n}\|\Delta_L\|_1 \le \frac{n^{3/2}(8n-2)\epsilon}{1-(12n+1)\epsilon}, \quad \text{with } \Delta_L := \tilde{L} - L. \tag{5.3}$$

If we write

$$\tilde{D} = D(I+W), \quad \text{with } W = \operatorname{diag}(w_1, w_2, \ldots, w_n),$$

then

$$\begin{aligned}\tilde{A} = \tilde{L}\tilde{D}\tilde{U} &= (L+\Delta_L)D(I+W)(U+\Delta_U) \\ &= (I+\Delta_L L^{-1})LD(U+\Delta_U+WU+W\Delta_U) \\ &= (I+E)A(I+F),\end{aligned}$$

where

$$E := \Delta_L L^{-1} \text{ , and} \tag{5.4}$$
$$F := U^{-1}\Delta_U + U^{-1}WU + U^{-1}W\Delta_U. \tag{5.5}$$

Since U is row diagonally dominant, we have $\|U^{-1}\|_{\max} = 1$ by (2.2). Then, from (5.1), (5.2), and (5.5), we get

$$\begin{aligned}\|F\|_{\max} &\le n\left(\|U^{-1}\|_{\max}\|\Delta_U\|_{\max} + \|U^{-1}\|_{\max}\|WU\|_{\max} + \|U^{-1}W\|_{\max}\|\Delta_U\|_{\max}\right) \\ &\le 3n^2\epsilon + \frac{2n^2\epsilon}{1-2n\epsilon} + \frac{2n^2\epsilon}{1-2n\epsilon}(3n\epsilon) \\ &= \frac{5n^2\epsilon}{1-2n\epsilon}.\end{aligned}$$

Thus, we have

$$\|F\|_2 \le \|F\|_F \le n\|F\|_{\max} \le \frac{5n^3\epsilon}{1-2n\epsilon} \le \nu.$$

Since L is column diagonally dominant, then $\|L^{-1}\|_{\max} = 1$, by (2.1), and hence $\|L^{-1}\|_2 \le n$. From (5.4) and (5.3), we obtain

$$\|E\|_2 \le \|\Delta_L\|_2\|L^{-1}\|_2 \le \frac{n^{3/2}(8n-2)\epsilon}{1-(12n+1)\epsilon}n = \nu.$$

So, if $0 \le \nu < 1$, then both $I+E$ and $I+F$ are nonsingular. Therefore, we can apply [14, Theorem 3.3] to obtain

$$|\tilde{\sigma}_i - \sigma_i| \le \gamma\sigma_i, \quad \text{for } i = 1, 2, \ldots, n, \tag{5.6}$$

where $\gamma = \max\{\|(I+E)(I+E)^T - I\|_2, \|(I+F)^T(I+F) - I\|_2\}$. Note that

$$\begin{aligned}\|(I+E)(I+E)^T - I\|_2 &= \|I+E+E^T+EE^T - I\|_2 = \|E+E^T+EE^T\|_2 \\ &\le \|E\|_2 + \|E^T\|_2 + \|E\|_2\|E^T\|_2 \le 2\|E\|_2 + \|E\|_2^2 \\ &\le 2\nu + \nu^2.\end{aligned}$$

Similarly, $\|(I+F)^T(I+F)-I\|_2 \leq 2\nu+\nu^2$. The theorem is proved by combining the last two inequalities with (5.6). $\square$

While the significant part of Theorem 5.1 is that the relative changes of the singular values are proportional to ϵ and are independent of any condition number, the provided bound is pessimistic since it contains the dimensional factor $n^{7/2}$. This is partly inherited from the perturbation bound for L, but it is also the result of bounding the 2-norm of various matrices and vectors from the max norm. As in the case of Theorem 4.4, we do not see how this pessimistic dimensional constant can be improved with the current approach.

Note that the bound (4.1) proved in [39] for symmetric positive semidefinite diagonally dominant matrices makes it natural to conjecture that the singular values satisfy the same perturbation bound, i.e., $|\tilde{\sigma}_i-\sigma_i| \leq \epsilon\sigma_i$, since the singular value problem is essentially a symmetric positive semidefinite eigenvalue problem. However, the matrices A and C in Example 2.6 show that this conjecture is false.

6. Bounds for eigenvalues of nonsymmetric matrices. The perturbation theory for the nonsymmetric eigenvalue problem is generally much more complex than for the symmetric eigenvalue problem. For example, general normwise perturbations of a nonsymmetric matrix A produce an absolute variation of the eigenvalues of A that may be much larger than the norm of the perturbation. The reason is that the absolute variation of each simple eigenvalue of A is governed by its Wilkinson condition number [9, p.149]. This condition number is eigenvalue dependent and is determined by the acute angle made by the left and the right eigenvectors of the eigenvalue, which is related to the departure from normality of A since for normal matrices the Wilkinson condition number is always equal to one. If A is a nonnormal row diagonally dominant matrix and we consider parameterized perturbations via diagonally dominant parts and off-diagonal entries, then the dependence of the eigenvalue variation on the Wilkinson condition number can still be expected; however, we will show in this section that the relative variation is independent of the magnitude of the eigenvalue itself.

Consider the classical analytic perturbation theory for the nonsymmetric eigenvalue problem (see [9, p.149]). Let λ be a simple eigenvalue of a general matrix $A \in \mathbb{R}^{n\times n}$ with a right eigenvector x and a left eigenvector y. The matrix $\tilde{A} = A+E$ has an eigenvalue $\tilde{\lambda}$ such that

$$\tilde{\lambda}-\lambda = \frac{y^*Ex}{y^*x} + \mathcal{O}\left(\|E\|_2^2\right) \tag{6.1}$$

and

$$|\tilde{\lambda}-\lambda| \leq \sec\theta(y,x)\|E\|_2 + \mathcal{O}\left(\|E\|_2^2\right), \tag{6.2}$$

where $\theta(y,x)$ is the acute angle between x and y, and $\sec\theta(y,x) = \dfrac{\|y\|_2\|x\|_2}{|y^*x|}$ is the Wilkinson condition number of the eigenvalue λ. The perturbation bound (6.2) concerns the absolute variation of the eigenvalue. The corresponding relative perturbation bound depends also on the magnitude of the eigenvalue itself as follows:

$$\frac{|\tilde{\lambda}-\lambda|}{|\lambda|} \leq \left(\sec\theta(y,x)\,\frac{\|A\|_2}{|\lambda|}\right)\frac{\|E\|_2}{\|A\|_2} + \mathcal{O}\left(\|E\|_2^2\right). \tag{6.3}$$

Observe that (6.3) shows that the relative variation of λ can be large compared to the relative size of the perturbation $\|E\|_2/\|A\|_2$ as a consequence of two facts: $\|A\|_2/|\lambda|$

can be large and/or $\sec\theta(y,x)$ can be large. For parameterized perturbations of row diagonally dominant matrices, we present in this section a new perturbation bound that removes the dependence on the magnitude of the eigenvalue, i.e., it removes the factor $\|A\|_2/|\lambda|$. We first present a modified version of (6.1) by using the left eigenvector $\tilde{y}$ of $\tilde{A}$.

LEMMA 6.1. *Let λ be an eigenvalue of $A \in \mathbb{R}^{n\times n}$ with a right eigenvector x and let $\tilde{\lambda}$ be an eigenvalue of $\tilde{A} = A+E$ with a left eigenvector $\tilde{y}$ such that $\tilde{y}^* x \neq 0$. Then,*

$$\tilde{\lambda} - \lambda = \frac{\tilde{y}^* E x}{\tilde{y}^* x} \tag{6.4}$$

and

$$|\tilde{\lambda} - \lambda| \leq \sec\theta(\tilde{y}, x)\|E\|_2. \tag{6.5}$$

Proof. Since $E = \tilde{A} - A$, we have

$$\tilde{y}^* E x = \tilde{y}^* \tilde{A} x - \tilde{y}^* A x = \left(\tilde{y}^* \tilde{\lambda}\right) x - \tilde{y}^* (\lambda x) = \left(\tilde{\lambda} - \lambda\right) \tilde{y}^* x,$$

from which (6.4) and hence (6.5) follow. □

Notice that (6.5) is very similar to (6.2); however, one advantage of (6.5) is that it is a straightforward inequality not containing asymptotically higher order error terms. On the other hand, (6.5) depends on the left eigenvector $\tilde{y}$ of $\tilde{A}$, which is not assumed to be known in a general setting. Interestingly, this turns out to be advantageous for our purpose as it will become evident in the proof of Theorem 6.2. We also note that in Lemma 6.1 neither λ nor $\tilde{\lambda}$ need to be simple eigenvalues and that $\tilde{\lambda}$ can be any eigenvalue of $\tilde{A}$, not necessarily the closest one to λ. However, for small perturbations E, if λ is not simple or $\tilde{\lambda}$ is not the eigenvalue approximating λ, then $\sec\theta(\tilde{y}, x)$ is expected to be extremely large and the bound (6.5) is not meaningful.

We now present in Theorem 6.2 a relative perturbation bound for eigenvalues of nonsymmetric row diagonally dominant matrices. We consider the general case of matrices with possibly both positive and negative diagonal entries and, therefore, the parametrization $A = \mathcal{D}(A_D, v, S)$ introduced in Definition 4.1 is used. Note that the perturbations considered in Theorem 6.2 preserve the signs of the diagonal entries. See the remarks before Theorem 4.4 concerning this assumption.

THEOREM 6.2. *Let $A = \mathcal{D}(A_D, v, S) \in \mathbb{R}^{n\times n}$ be such that $v \geq 0$ and let λ be an eigenvalue of A with a right eigenvector x. Let $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}, S) \in \mathbb{R}^{n\times n}$ be such that*

$$|\tilde{v} - v| \leq \epsilon v \quad \text{and} \quad |\tilde{A}_D - A_D| \leq \epsilon |A_D|, \quad \text{for some } 0 \leq \epsilon < 1,$$

and let $\tilde{\lambda}$ be an eigenvalue of $\tilde{A}$ with a left eigenvector $\tilde{y}$ such that $\tilde{y}^* x \neq 0$. If $(13n + 7n^3 \sec\theta(\tilde{y}, x))\,\epsilon < 1$, then

$$|\tilde{\lambda} - \lambda| \leq \frac{8n^{7/2} + 7n^3}{1 - (13n + 7n^3 \sec\theta(\tilde{y}, x))\,\epsilon} \sec\theta(\tilde{y}, x)\epsilon\,|\lambda|\,, \tag{6.6}$$

where $\sec\theta(\tilde{y}, x) = \dfrac{\|\tilde{y}\|_2 \|x\|_2}{|\tilde{y}^* x|}$.

Proof. Clearly SA and $S\tilde{A}$ are row diagonally dominant with nonnegative diagonal entries and they satisfy condition (2.3) of Theorem 2.4. Without loss of generality, we assume that SA is arranged for column diagonal dominance pivoting. Then SA has LDU factorization $SA = LDU$ with L being column diagonally dominant and U being row diagonally dominant. By (2.1-2.2), we have $\|L^{-1}\|_2 \le n\|L^{-1}\|_{\max} \le n$, $\|U\|_2 \le \sqrt{n}\|U\|_\infty \le 2\sqrt{n}$, and $\|U^{-1}\|_2 \le n\|U^{-1}\|_{\max} \le n$.

It follows from Theorem 2.4 that $S\tilde{A}$ has LDU factorization $S\tilde{A} = \tilde{L}\tilde{D}\tilde{U}$ and

$$|\Delta_D| \le \frac{2n\epsilon}{1-2n\epsilon}D, \quad \text{with } \Delta_D := \tilde{D} - D, \tag{6.7}$$

$$\|\Delta_U\|_2 \le n\|\Delta_U\|_{\max} \le 3n^2\epsilon, \quad \text{with } \Delta_U := \tilde{U} - U, \tag{6.8}$$

and

$$\|\Delta_L\|_2 \le \sqrt{n}\|\Delta_L\|_1 \le \frac{n^{3/2}(8n-2)}{1-(12n+1)\epsilon}\epsilon, \quad \text{with } \Delta_L := \tilde{L} - L. \tag{6.9}$$

We write $E := \tilde{A} - A = S(\tilde{L}\tilde{D}\tilde{U} - LDU)$ as

$$E = S\Delta_L DU + S\tilde{L}\Delta_D U + S\tilde{L}\tilde{D}\Delta_U\,.$$

Combining this expression for E with Lemma 6.1, we obtain

$$\begin{aligned}(\tilde{\lambda} - \lambda)(\tilde{y}^* x) &= \tilde{y}^* S\Delta_L DUx + \tilde{y}^* S\tilde{L}\Delta_D Ux + \tilde{y}^* S\tilde{L}\tilde{D}\Delta_U x \\ &= \lambda\tilde{y}^* S\Delta_L L^{-1}Sx + \tilde{\lambda}\tilde{y}^*\tilde{U}^{-1}\tilde{D}^\dagger\Delta_D Ux + \tilde{\lambda}\tilde{y}^*\tilde{U}^{-1}\Delta_U x,\end{aligned} \tag{6.10}$$

where $\tilde{D}^\dagger$ is the Moore-Penrose pseudo-inverse of $\tilde{D}$ and we have used $DUx = L^{-1}SAx = \lambda L^{-1}Sx$, $\tilde{y}^*S\tilde{L}\tilde{D} = \tilde{y}^*\tilde{A}\tilde{U}^{-1} = \tilde{\lambda}\tilde{y}^*\tilde{U}^{-1}$, and $\tilde{y}^*S\tilde{L}\Delta_D = \tilde{y}^*S\tilde{L}(\tilde{D}\tilde{D}^\dagger)\Delta_D = \tilde{\lambda}\tilde{y}^*\tilde{U}^{-1}\tilde{D}^\dagger\Delta_D$. In addition, note that $S\tilde{A}$ is also row diagonally dominant, since $\tilde{v} \ge 0$, and, so, $\tilde{U}$ is row diagonally dominant. This implies, by (2.2), that $\|\tilde{U}^{-1}\|_2 \le n\|\tilde{U}^{-1}\|_{\max} \le n$. With this bound and (6.7), (6.8), and (6.9), we get

$$\|\Delta_L L^{-1}\|_2 \le \|\Delta_L\|_2\|L^{-1}\|_2 \le \frac{n^{5/2}(8n-2)}{1-(12n+1)\epsilon}\epsilon,$$

$$\|\tilde{D}^\dagger\Delta_D\|_2 = \|(I + D^\dagger\Delta_D)^{-1}D^\dagger\Delta_D\|_2 \le \|(I + D^\dagger\Delta_D)^{-1}\|_2\,\|D^\dagger\Delta_D\|_2 \le \frac{2n\epsilon}{1-4n\epsilon},$$

and

$$\|\tilde{U}^{-1}\Delta_U\|_2 \le \|\tilde{U}^{-1}\|_2\,\|\Delta_U\|_2 \le 3n^3\epsilon.$$

Substituting these into (6.10), we obtain

$$\begin{aligned}|\tilde{\lambda} - \lambda||\tilde{y}^* x| &\le \|\tilde{y}\|_2\|x\|_2\left(|\lambda|\frac{n^{5/2}(8n-2)\epsilon}{1-(12n+1)\epsilon} + |\tilde{\lambda}|\,n\,\frac{2n\epsilon}{1-4n\epsilon}\,2\sqrt{n} + |\tilde{\lambda}|\,3n^3\epsilon\right) \\ &\le \|\tilde{y}\|_2\|x\|_2\left(|\lambda|\frac{8n^{7/2}\epsilon}{1-13n\epsilon} + |\tilde{\lambda}|\frac{7n^3\epsilon}{1-4n\epsilon}\right)\end{aligned}$$

and thus

$$|\tilde{\lambda} - \lambda| \le \sec\theta(\tilde{y}, x)\left(|\lambda|\frac{8n^{7/2}\epsilon}{1-13n\epsilon} + |\tilde{\lambda}|\frac{7n^3\epsilon}{1-13n\epsilon}\right).$$

Finally, use $|\tilde\lambda| \le |\tilde\lambda - \lambda| + |\lambda|$ and rearrange the inequality above to produce the desired result. ☐

Theorem 6.2 improves the classical bound (6.3) in that the relative perturbation $|\tilde\lambda - \lambda|/|\lambda|$ in the eigenvalue is proportional to $\sec\theta(\tilde y, x)\epsilon$ but independent of the eigenvalue itself. In addition, we have a rigorous inequality independent of any high order term. A drawback of Theorem 6.2 is that the condition number $\sec\theta(\tilde y, x)$ is defined from the left eigenvector of $\tilde A$. However, if $\tilde\lambda$ approximates a simple eigenvalue λ, then $\tilde y \approx y$ and $\sec\theta(\tilde y, x) \approx \sec\theta(y, x)$ in an asymptotic sense. In addition, as discussed just after Lemma 6.1, this is the only situation in which Theorem 6.2 is really useful and meaningful, since otherwise $\tilde y^* x \approx 0$ and it renders a meaningless bound. Therefore, Theorem 6.2 implicitly requires that $\tilde\lambda$ is an eigenvalue of $\tilde A$ that approximates a simple eigenvalue λ of A.

Theorem 6.2 can be generalized to a structured perturbation problem where a general matrix, i.e., not necessarily diagonally dominant, is perturbed via small changes in the factors of a rank-revealing decomposition of the matrix. Recall that given a matrix $A \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A) = r$, we say that $A = XDY \in \mathbb{R}^{m\times n}$ is a rank-revealing decomposition of A [8], if $D \in \mathbb{R}^{r\times r}$ is diagonal and nonsingular, and $X \in \mathbb{R}^{m\times r}$ and $Y \in \mathbb{R}^{r\times n}$ are well-conditioned matrices. Since X and Y may be rectangular matrices, their condition numbers are defined via their Moore-Penrose pseudo-inverses, denoted by $X^\dagger$ and $Y^\dagger$, as $\kappa_2(X) = \|X\|_2\,\|X^\dagger\|_2$ and $\kappa_2(Y) = \|Y\|_2\,\|Y^\dagger\|_2$. Next, we consider in Theorem 6.3 perturbations of A obtained via small entrywise relative perturbations in the diagonal factor D and small normwise relative perturbations in the factors X and Y. The bound in Theorem 6.3 may be applicable to some special matrices for which a rank-revealing decomposition can be accurately computed [8, 12, 13].

THEOREM 6.3. *Let $A = XDY \in \mathbb{R}^{n\times n}$ be a rank-revealing decomposition and let $\tilde A = \tilde X \tilde D \tilde Y \in \mathbb{R}^{n\times n}$ be such that*

$$\tilde X = X + \Delta_X, \quad \tilde D = D + \Delta_D, \quad \tilde Y = Y + \Delta_Y$$

with

$$|\Delta_D| \le \epsilon|D|, \quad \|\Delta_X\|_2 \le \epsilon\|X\|_2, \text{ and } \|\Delta_Y\|_2 \le \epsilon\|Y\|_2 \tag{6.11}$$

for some $0 \le \epsilon < 1$. Let λ be an eigenvalue of A with right eigenvector x and let $\tilde\lambda$ be an eigenvalue of $\tilde A$ with left eigenvector $\tilde y$ such that $\tilde y^* x \ne 0$. Let $\kappa = \max\{\kappa_2(X), \kappa_2(Y)\}$. If $\epsilon\,\kappa\,(1 + \sec\theta(\tilde y, x)) < 1$, then

$$|\tilde\lambda - \lambda| \le \epsilon\,\kappa \sec\theta(\tilde y, x)\,\frac{3+\epsilon}{1 - \epsilon\,\kappa\,(1 + \sec\theta(\tilde y, x))}\,|\lambda|\,. \tag{6.12}$$

Proof. Let $r = \operatorname{rank}(A)$. So $X^\dagger X = I_r$ and $YY^\dagger = I_r$, since XDY is a rank-revealing decomposition. In addition, note that[1] $\operatorname{rank}(X) = \operatorname{rank}(\tilde X) = r$ and $\operatorname{rank}(Y) = \operatorname{rank}(\tilde Y) = r$. So $\tilde X^\dagger \tilde X = I_r$ and $\tilde Y \tilde Y^\dagger = I_r$ also hold.

Observe that

$$\tilde A - A = \Delta_X DY + \tilde X \Delta_D Y + \tilde X \tilde D \Delta_Y\,.$$

[1] These two equalities follow from (6.11). Let us prove it only for X, since it is is similar for Y. According to Weyl perturbation theorem [36] for singular values, we have $|\sigma_i(\tilde X) - \sigma_i(X)| \le \|\Delta_X\|_2 \le \epsilon\|X\|_2$, for $i = 1, \ldots, r$, where $\sigma_i(X)$ and $\sigma_i(\tilde X)$ are the singular values of X and $\tilde X$ respectively arranged in decreasing order. So $|\sigma_i(\tilde X) - \sigma_i(X)|/\sigma_i(X) \le \epsilon\kappa_2(X) < 1$, for $i = 1, \ldots, r$. This and $\sigma_i(X) \ne 0$ imply that $\sigma_i(\tilde X) \ne 0$ for all i.

Applying (6.4) yields

$$(\tilde{y}^* x)\left(\tilde{\lambda} - \lambda\right) = \tilde{y}^* \Delta_X D Y x + \tilde{y}^* \tilde{X} \Delta_D Y x + \tilde{y}^* \tilde{X} \tilde{D} \Delta_Y x$$
$$= \lambda \tilde{y}^* \Delta_X X^\dagger x + \lambda \tilde{y}^* \tilde{X} \Delta_D D^{-1} X^\dagger x + \tilde{\lambda} \tilde{y}^* \tilde{Y}^\dagger \Delta_Y x, \qquad (6.13)$$

since $DYx = X^\dagger A x = \lambda X^\dagger x$, $\Delta_D Y x = \Delta_D (D^{-1} D) Y x = \lambda \Delta_D D^{-1} X^\dagger x$, and $\tilde{y}^* \tilde{X} \tilde{D} = \tilde{y}^* \tilde{A} \tilde{Y}^\dagger = \tilde{\lambda} \tilde{y}^* \tilde{Y}^\dagger$. From the assumption (6.11), we get

$$\|\Delta_X X^\dagger\|_2 \leq \|\Delta_X\|_2 \, \|X^\dagger\|_2 \leq \epsilon \, \kappa_2(X), \qquad (6.14)$$

$$\|\tilde{X} \Delta_D D^{-1} X^\dagger\|_2 \leq \|X + \Delta_X\|_2 \, \|\Delta_D D^{-1}\|_2 \|X^\dagger\|_2 \leq \epsilon (1 + \epsilon) \, \kappa_2(X), \qquad (6.15)$$

and, if $\tilde{\sigma}_r$ and σ_r are, respectively, the smallest singular values of $\tilde{Y}$ and Y,

$$\|\tilde{Y}^\dagger \Delta_Y\|_2 \leq \frac{\|\Delta_Y\|_2}{\tilde{\sigma}_r} \leq \frac{\epsilon \, \|Y\|_2}{\sigma_r - \|\Delta_Y\|_2} \leq \frac{\epsilon \, \|Y\|_2}{\sigma_r - \epsilon \, \|Y\|_2} \leq \frac{\epsilon \, \kappa_2(Y)}{1 - \epsilon \, \kappa_2(Y)}. \qquad (6.16)$$

Combining (6.14)-(6.15)-(6.16) with (6.13), we have

$$|\tilde{y}^* x| \, |\tilde{\lambda} - \lambda| \leq \|\tilde{y}\|_2 \|x\|_2 \left(|\lambda| \, \epsilon \, (2 + \epsilon) \, \kappa_2(X) + |\tilde{\lambda}| \, \frac{\epsilon \, \kappa_2(Y)}{1 - \epsilon \, \kappa_2(Y)} \right).$$

Finally, use $|\tilde{\lambda}| \leq |\tilde{\lambda} - \lambda| + |\lambda|$ and rearrange the inequality above to obtain (6.12). ∎

We finish this section with some remarks on other possible strategies for obtaining relative perturbation bounds for eigenvalues of nonsymmetric matrices when they are perturbed via a rank-reveling decomposition as in (6.11). This type of perturbations of rank-revealing decompositions can always be written as a *multiplicative* perturbation of the original matrix. This has been used before in [4, 8, 12, 13]. Then, it is possible to use relative bounds for eigenvalues of nonsymmetric matrices under multiplicative perturbations which are already available in the literature [23, Section 5] (see also the original references [15] and [26]). Essentially, two type of relative bounds can be found: Bauer-Fike and Hoffman-Wielandt bounds. The Bauer-Fike bounds require A to be diagonalizable and depend on the condition number of the whole eigenvector matrix of A, that is, the square matrix whose columns are all the eigenvectors of A. The Hoffman-Wielandt bounds still require stronger assumptions, since they require both A and $\tilde{A}$ to be diagonalisable and they depend on the product of the condition numbers of *both* the whole eigenvector matrices of A and $\tilde{A}$. The main drawback of these bounds is that the condition number of the whole eigenvector matrix is larger than the largest Wilkinson condition number of all the individual eigenvalues [9, Theorem 4.7]. Thus, the relative Bauer-Fike and Hoffman-Wielandt bounds may be very pessimistic in situations where only some eigenvalues have large Wilkinson condition numbers, but the condition numbers of other eigenvalues are moderate. Theorem 6.3 presented here has the obvious advantage of depending essentially only on the Wilkinson condition number of each individual eigenvalue and, in addition, it does not require that the matrix A be diagonalizable.

7. Concluding remarks. We have systematically studied the relative perturbation theory for row diagonally dominant matrices under small componentwise perturbations of their diagonally dominant parts and off-diagonal entries. The use of this parameterized perturbation has been the key to derive strong relative perturbation bounds for inverses, solution to linear systems, the symmetric indefinite eigenvalue problem, the singular value problem, and the nonsymmetric eigenvalue problem. These bounds demonstrate that potentially much more accurate algorithms than the traditional ones are possible for solving all these problems via the use of diagonally dominant parts and off-diagonal entries. Indeed, such high relative accuracy algorithms have already been obtained for the LDU factorization and the singular value problem in [38], and the results in the present paper show that highly accurate algorithms for other problems can be also obtained by combining the LDU algorithm in [38] with the algorithms in [4, 12, 13]. One challenging open problem in this area is to develop algorithms to compute the eigenvalues of nonsymmetric diagonally dominant matrices with the relative accuracy determined by Theorem 6.2. This will be the subject of future research.

REFERENCES

[1] A. S. Alfa, J. Xue, and Q. Ye, *Entrywise perturbation theory for diagonally dominant M-matrices with applications*, Numer. Math., 90 (2002), pp. 401–414.

[2] S. Bora, *Structured eigenvalue condition number and backward error of a class of polynomial eigenvalue problems*, SIAM J. Matrix Anal. Appl., 31 (2009), pp. 900–917.

[3] S. Bora and V. Mehrmann, *Linear perturbation theory for structured matrix pencils arising in control theory*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 148–169.

[4] N. Castro-González, J. Ceballos, F. M. Dopico, and J. M. Molera, *Accurate solution of structured least squares problems via rank-revealing decompositions*, SIAM J. Matrix Anal. Appl., 34 (2013), pp. 1112–1128.

[5] T. F. Chan and D. E. Foulser, *Effectively well-conditioned linear systems*, SIAM J. Sci. Statist. Comput., 9 (1988), pp. 963–969.

[6] X.-W. Chang and R.-C. Li, *Multiplicative perturbation analysis for QR factorizations*, Numer. Algebra Control Optim., 1 (2011), pp. 301–316.

[7] M. Dailey, F. M. Dopico, and Q. Ye, *New relative perturbation bounds for LDU factorizations of diagonally dominant matrices*. Submitted (available at `http://gauss.uc3m.es/web/personal_web/fdopico/index_sp.html`).

[8] J. Demmel, M. Gu, S. Eisenstat, I. Slapničar, K. Veselić, and Z. Drmač, *Computing the singular value decomposition with high relative accuracy*, Linear Algebra Appl., 299 (1999), pp. 21–80.

[9] J. W. Demmel, *Applied Numerical Linear Algebra*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1997.

[10] I. S. Dhillon and B. N. Parlett, *Orthogonal eigenvectors and relative gaps*, SIAM J. Matrix Anal. Appl., 25 (2003), pp. 858–899.

[11] F. M. Dopico and P. Koev, *Perturbation theory for the LDU factorization and accurate computations for diagonally dominant matrices*, Numer. Math., 119 (2011), pp. 337–371.

[12] F. M. Dopico, P. Koev, and J. M. Molera, *Implicit standard Jacobi gives high relative accuracy*, Numer. Math., 113 (2009), pp. 519–553.

[13] F. M. Dopico and J. M. Molera, *Accurate solution of structured linear systems via rank-revealing decompositions*, IMA J. Numer. Anal., 32 (2012), pp. 1096–1116.

[14] S. C. Eisenstat and I. C. F. Ipsen, *Relative perturbation techniques for singular value problems*, SIAM J. Numer. Anal., 32 (1995), pp. 1972–1988.

[15] S. C. Eisenstat and I. C. F. Ipsen, *Relative perturbation results for eigenvalues and eigenvectors of diagonalisable matrices*, BIT, 38 (1998), pp. 502–509.

[16] C. Ferreira, B. Parlett, and F. M. Dopico, *Sensitivity of eigenvalues of an unsymmetric tridiagonal matrix*, Numer. Math., 122 (2012), pp. 527–555.

[17] G. Golub and C. Van Loan, *Matrix Computations*, Johns Hopkins University Press, Baltimore, MD, 3rd ed., 1996.

[18] D. J. Higham and N. J. Higham, *Backward error and condition of structured linear systems*, SIAM J. Matrix Anal. Appl., 13 (1992), pp. 162–175.

[19] ———, *Structured backward error and condition of generalized eigenvalue problems*, SIAM J. Matrix Anal. Appl., 20 (1999), pp. 493–512.

[20] N. J. Higham, *Accuracy and Stability of Numerical Algorithms*, Society for Industrial and

Applied Mathematics (SIAM), Philadelphia, PA, second ed., 2002.
[21] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.
[22] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1994. Corrected reprint of the 1991 original.
[23] I. C. F. Ipsen, *Relative perturbation results for matrix eigenvalues and singular values*, in Acta Numerica, 1998, vol. 7 of Acta Numer., Cambridge Univ. Press, Cambridge, 1998, pp. 151–201.
[24] M. Karow, D. Kressner, and F. Tisseur, *Structured eigenvalue condition numbers*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 1052–1068.
[25] P. Koev, *Accurate eigenvalues and SVDs of totally nonnegative matrices*, SIAM J. Matrix Anal. Appl., 27 (2005), pp. 1–23.
[26] R.-C. Li, *Relative perturbation theory. III. More bounds on eigenvalue variation*, Linear Algebra Appl., 266 (1997), pp. 337–345.
[27] ———, *A bound on the solution to a structured Sylvester equation with an application to relative perturbation theory*, SIAM J. Matrix Anal. Appl., 21 (1999), pp. 440–445.
[28] R.-C. Li, Y. Nakatsukasa, N. Truhar, and S. Xu, *Perturbation of partitioned Hermitian definite generalized eigenvalue problems*, SIAM J. Matrix Anal. Appl., 32 (2011), pp. 642–663.
[29] A. M. Ostrowski, *Note on bounds for determinants with dominant principal diagonal*, Proc. Amer. Math. Soc., 3 (1952), pp. 26–30.
[30] B. N. Parlett, *Spectral sensitivity of products of bidiagonals*, Linear Algebra Appl., 275/276 (1998), pp. 417–431.
[31] ———, *Perturbation of eigenpairs of factored symmetric tridiagonal matrices*, Found. Comput. Math., 3 (2003), pp. 207–223.
[32] ———, *A bidiagonal matrix determines its hyperbolic SVD to varied relative accuracy*, SIAM J. Matrix Anal. Appl., 26 (2005), pp. 1022–1057.
[33] J. M. Peña, *LDU decompositions with L and U well conditioned*, Electron. Trans. Numer. Anal., 18 (2004), pp. 198–208.
[34] S. M. Rump, *Eigenvalues, pseudospectrum and structured perturbations*, Lin. Alg. Appl., 413 (2006), pp. 567–593.
[35] P. N. Shivakumar, J. J. Williams, Q. Ye, and C. A. Marinov, *On two-sided bounds related to weakly diagonally dominant M-matrices with application to digital circuit dynamics*, SIAM J. Matrix Anal. Appl., 17 (1996), pp. 298–312.
[36] G. W. Stewart and J.-G. Sun, *Matrix Perturbation Theory*, Academic Press, New York, 1990.
[37] J. M. Varah, *A lower bound for the smallest singular value of a matrix*, Linear Algebra Appl., 11 (1975), pp. 3–5.
[38] Q. Ye, *Computing singular values of diagonally dominant matrices to high relative accuracy*, Math. Comp., 77 (2008), pp. 2195–2230.
[39] Q. Ye, *Relative perturbation bounds for eigenvalues of symmetric positive definite diagonally dominant matrices*, SIAM J. Matrix Anal. Appl., 31 (2009), pp. 11–17.