NEW RELATIVE PERTURBATION BOUNDS FOR LDU FACTORIZATIONS OF DIAGONALLY DOMINANT MATRICES

MEGAN DAILEY*, FROILÁN M. DOPICO†, AND QIANG YE‡

Abstract. This work introduces new relative perturbation bounds for the LDU factorization of (row) diagonally dominant matrices under structure-preserving componentwise perturbations. These bounds establish that if (row) diagonally dominant matrices are parameterized via their diagonally dominant parts and off-diagonal entries, then tiny relative componentwise perturbations of these parameters produce tiny relative normwise variations of the L and U factors and tiny relative entrywise variations of the factor D. These results improve previous bounds in an essential way, by including LDU factorizations computed via the column diagonal dominance pivoting strategy. This strategy is specific for (row) diagonally dominant matrices and has the key advantage of yielding L and U factors which are guaranteed to be well-conditioned and, so, the corresponding LDU factorization is guaranteed to be a rank-revealing decomposition. Since rank-revealing decompositions play a fundamental role in highly accurate matrix computations, the results presented in this paper have some important implications, because they will allow us to prove rigorously in a follow-up work that most of the standard tasks in numerical linear algebra can be performed with guaranteed high accuracy for the relevant class of diagonally dominant matrices.

Key words. accurate computations, column diagonal dominance pivoting, diagonally dominant matrices, diagonally dominant parts, LDU factorization, rank-revealing decomposition, relative perturbation theory

AMS subject classifications. 65F05, 65F15, 15A18, 15A23, 15B99

1. Introduction.

Perturbation analysis is a fundamental topic in numerical linear algebra that determines the accuracy to which a given problem can be solved numerically [29]. Classical perturbation theory considers mostly unstructured normwise or entrywise perturbations of matrices [19, 20]. However, in the last two decades, the interests of matrix perturbation theory have expanded and many works have been focused on deriving sharper perturbation bounds when structured perturbations of important classes of structured matrices are considered (see, for instance, [1, 3, 4, 6, 11, 17, 18, 22, 23, 24, 25, 27, 28, 31, 32, 34] and the references therein). In this paper, we study perturbation bounds for LDU factorizations of diagonally dominant matrices under structure-preserving perturbations. Here, $A = LDU$ is an LDU factorization of A if L is a unit lower triangular matrix, D is a diagonal matrix, and U is a unit upper triangular matrix.

The LDU factorization, or the LU factorization, is one of the most important matrix factorizations and has many applications, such as solving systems of linear equations, inverting matrices, and computing determinants [16]. Depending on its uses, a pivoting scheme is usually employed to produce a factorization with certain desirable properties. For example, partial pivoting is the standard choice to compute a backward stable LDU factorization and to solve systems of linear equations [20]. For some classes of matrices, it is not necessary to use any pivoting strategy to produce a backward stable LDU factorization. Diagonally dominant matrices are among the

*Indiana University Kokomo 2300 S. Washington St., Kokomo, IN 46904-9003, USA. E-mail: medailey@iuk.edu. Research supported in part by NSF under grant DMS-1318633.

†Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM and Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain. E-mail: dopico@math.uc3m.es. Research supported in part by *Ministerio de Economía y Competitividad* of Spain under grant MTM2012-32542.

‡Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA. E-mail: qiang.ye@uky.edu. Research supported in part by NSF under grant DMS-1318633.

most important of these classes, since they arise in many applications and have very favorable numerical properties [16, 20]. They are the matrices considered in this paper. For brevity, we will focus on *row* diagonally dominant matrices, although the results we present hold for column diagonally dominant matrices with obvious modifications or just by taking transposes. For these matrices, the LDU factorization without pivoting has another interesting property: given any $n \times n$ row diagonally dominant matrix, the U matrix in its LDU factorization inherits the row diagonal dominance property. As a result, U is well-conditioned with $\kappa_\infty(U) \leq 2n$ [26]. In contrast, L may not be well-conditioned and, since this property may be essential in the use of the LDU factorization (e.g. deriving some more accurate algorithms to be discussed below), it is important to study possible pivoting strategies that preserve row diagonal dominance and compute a well-conditioned factor L as well. The first candidate is the complete (diagonal) pivoting [20]. In this case, all entries of L are bounded in absolute values by 1 and, as a consequence, L is almost always well-conditioned, but this is not theoretically guaranteed. It turns out that a slightly more expensive pivoting scheme introduced in [26, 33], valid only for row diagonally dominant matrices and some other structured matrices, will produce a column diagonally dominant matrix L, which is then guaranteed to be well-conditioned with $\kappa_\infty(L) \leq n^2$ and $\kappa_1(L) \leq 2n$ [26]. This pivoting strategy is called here *column diagonal dominance pivoting.*

We are interested in an LDU factorization with both L and U well-conditioned because in this case it becomes a *rank-revealing decomposition* (RRD); see [9]. RRDs are key components in high relative accuracy algorithms for computing quantities of different magnitudes, where *high relative accuracy* means that these algorithms produce relative forward errors in the order of unit roundoff even when they are applied to extremely ill-conditioned problems for which standard algorithms may fail to produce a single digit of accuracy. For example, two algorithms are presented in [9] to compute the singular value decomposition of a matrix A to high relative accuracy if an *accurate* RRD $A = XDY$ is available, in the sense that X and Y are well-conditioned and have been computed with tiny normwise relative errors, and D is diagonal and has been computed with tiny entrywise relative errors. Accurate RRDs are also utilized in [10, 12] to develop algorithms that compute eigenvalues and eigenvectors of symmetric matrices with high relative accuracy. In addition, accurate RRDs are used in [5] and [14] to compute solutions to structured linear systems and least squares problems which are much more accurate than those computed by standard algorithms. All of these works depend critically on computing an *accurate* RRD of A first, which is a difficult problem that can be solved only for some special structured matrices via new algorithms carefully designed to exploit the corresponding structures. See [9, 12, 14] for a detailed account of many classes of structured matrices for which it is possible to compute accurate RRDs. Another essential feature of these new highly accurate algorithms in [5, 9, 12, 14] is that the forward error bounds depend on the condition numbers of the well-conditioned factors X and Y but not on the condition number of the matrix A.

As discussed above, the LDU factorization is guaranteed to be a RRD for row diagonally dominant matrices if the column diagonal dominance pivoting is used. Moreover, in practice, the complete diagonal pivoting is almost always sufficient for computing a rank-revealing LDU factorization. Furthermore, if standard algorithms for the LDU factorization, combined with either of these two pivoting strategies, are applied in finite precision arithmetic to a row diagonally dominant matrix A, then they compute LDU factorizations which are RRDs and are backward stable with respect to A, but unfortunately, *this is not enough for producing accurate L, D, and U factors and hence accurate RRDs.* The reason is that classic perturbation theory for the LU factorization establishes that the entries of A do not determine accurately its L, D, and U factors (see [2, 7, 13, 30] and [20, p. 194]).

In this context, a structured algorithm for computing the LDU factorization of row diagonally dominant matrices has been developed recently in [33]. This algorithm works for both the column diagonal dominance pivoting and the complete diagonal pivoting, and it computes accurate L and U factors with relative normwise errors of the order of unit roundoff, and accurate D factors with relative entrywise errors of the order of unit roundoff. This is supported by many numerical experiments and also by a direct forward error analysis in [33]. Unfortunately, the error bound involves very large dimensional constants that question the accuracy of the computed factors, although it has the remarkable property of being independent of any condition number. Motivated by this fact, a structured perturbation theory for the LDU factorization of row diagonally dominant matrices has been derived in [11], which has led to a much improved error analysis for the algorithm in [33] involving only dimensional constants of moderate size. However, the error bound on L in [11] is guaranteed to be small only under the assumption that the complete diagonal pivoting is used. Since the complete diagonal pivoting does not lead to a guaranteed well-conditioned factor L, a fundamental question is whether or not a strong perturbation bound for L still holds when the column diagonal dominance pivoting is used, for which the LDU factorization is rigorously a RRD.

In this paper, we extend the perturbation theory in [11] to allow the fundamental assumption that the column diagonal dominance pivoting is used in defining the LDU factorization. This generalization will prove to be crucial in a separate work [8] that systematically studies structured perturbation properties of diagonally dominant matrices for many other linear algebra problems. It is also central to show that the combination of the algorithm for the LDU factorization in [33] with those in [5, 9, 12, 14] allows us to perform many matrix computations efficiently and very accurately on diagonally dominant matrices, with forward relative errors in the order of unit roundoff and independent of the condition numbers of any of the factors of the involved RRDs. The development of the new perturbation bounds requires considerable technical efforts and a number of substantially new ideas over the techniques used in [11].

We emphasize that the key idea underlying these strong perturbation properties is the need to parameterize any row diagonally dominant matrix by its off-diagonal entries and diagonally dominant parts. This parametrization was introduced in [33] and used in [34] to derive relative perturbation bounds for eigenvalues of symmetric positive semidefinite diagonally dominant matrices. It was also essential in [11] and recently has led to very strong perturbation bounds for many other problems [8]. As pointed out in [34], this parametrization often corresponds to physical parameters and is natural in many applications of diagonal dominant matrices.

The rest of this paper is organized as follows. In section 2, we give an overview of diagonally dominant matrices and revise previous results that are needed in this work. Section 3 presents the new perturbation bounds for the LDU factorization and their proofs. Finally, in section 4, conclusions and lines of future research are discussed. Before proceeding, we present below the notation used in this paper.

NOTATION: In this paper we consider only real matrices and we denote by $\mathbb{R}^{m\times n}$ the set of $m \times n$ real matrices. The entries of a matrix A are a_{ij} and $|A|$ is the matrix with entries $|a_{ij}|$. The inequality $A \geq B$ for matrices means $a_{ij} \geq b_{ij}$ for all i, j, and the inequality $v \geq w$ for vectors means $v_i \geq w_i$ for all i. Analogously, the inequality $v \geq 0$ for the vector v means $v_i \geq 0$ for all i. We use the MATLAB notation for submatrices. That is, $A(i : j, k : l)$ denotes the submatrix of A formed by rows i through j and columns k through l. We use $A(i', j')$ to denote the submatrix of A formed by deleting row i and column j from A. Let $\alpha = [i_1, i_2, \ldots, i_p]$, where $1 \leq i_1 < i_2 < \cdots < i_p \leq m$, and $\beta = [j_1, j_2, \ldots, j_q]$, where $1 \leq j_1 < j_2 < \cdots < j_q \leq n$. Then $A(\alpha, \beta)$ denotes the submatrix of A that consists of rows $i_1, i_2, \ldots, i_p$ and columns $j_1, j_2, \ldots, j_q$. In MATLAB notation, $1 : k$ denotes the row vector $[1, 2, \ldots, k]$. For convenience, we also use the notation $1 : k$ to denote the set $\{1, 2, \ldots, k\}$. We denote by I_s the $s \times s$ identity matrix and by 0_s the $s \times s$ zero matrix. Two matrix norms will be used: $\|A\|_1 = \max_j \sum_i |a_{ij}|$ and $\|A\|_\infty = \max_i \sum_j |a_{ij}|$. The condition numbers of a nonsingular matrix A in any of these norms is denoted as $\kappa_i(A) := \|A\|_i \, \|A^{-1}\|_i$, for $i = 1, \infty$. The sign of $x \in \mathbb{R}$ is $\operatorname{sign}(x)$, where $\operatorname{sign}(0)$ is defined to be 1.

2. Preliminaries. In this section, we give an overview of diagonally dominant matrices and some of their properties. More information on this topic can be found in [11, Sec. 2] and [20, 21].

DEFINITION 2.1. *A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is said to be row diagonally dominant if $|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|$ for $i = 1, \ldots, n$.*

In the rest of the paper, all row diagonally dominant matrices $A \in \mathbb{R}^{n\times n}$ that are considered satisfy $a_{ii} \geq 0$ for $i = 1, \ldots, n$. This does not impose any restriction for studying LDU factorizations, since we can multiply A by a diagonal matrix S with diagonal entries equal to ± 1 to get this property, and the LDU factorizations of A and SA are trivially related each other.

An idea that has played a key role in deriving relative perturbation bounds and high relative accuracy algorithms for row diagonally dominant matrices [33, 34, 11] is to reparameterize these matrices in terms of their diagonally dominant parts and off-diagonal entries as follows.

DEFINITION 2.2. *Given a matrix $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ and a vector $v = [v_i] \in \mathbb{R}^n$, we use $\mathcal{D}(M, v)$ to denote the matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ whose off-diagonal entries are the same as those of M and whose ith diagonal entry is $a_{ii} = v_i + \sum_{j\neq i} |m_{ij}|$, for $i = 1, \ldots, n$. Namely, we write*

$$A = \mathcal{D}(M, v)$$

and call it the representation of A by diagonally dominant parts v and off-diagonal entries m_{ij}, $i \neq j$, if

$$a_{ij} = m_{ij}, \text{ for } i \neq j, \quad \text{and} \quad a_{ii} = v_i + \sum_{j\neq i} |m_{ij}|, \text{ for } i = 1, \ldots, n.$$

Definition 2.2 constructs A from diagonally dominant parts and off-diagonal entries. The converse process is obvious: given a matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, we denote by $A_D \in \mathbb{R}^{n\times n}$ the matrix whose off-diagonal entries are the same as those of A and whose diagonal entries are zero. Then, letting $v_i = a_{ii} - \sum_{j\neq i} |a_{ij}|$, $i = 1, \ldots, n$, and $v = [v_1, v_2, \ldots, v_n]^T$, we have

$$A = \mathcal{D}(A_D, v) \tag{2.1}$$

as the representation of A by diagonally dominant parts and off-diagonal entries.

Observe that in the representation $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ the condition $v \geq 0$ is equivalent to the statement that A is row diagonally dominant and has nonnegative diagonal entries. We emphasize that the condition $v \geq 0$ will be used as an assumption in most of the results in section 3 and, so, the reader should bear in mind its underlying meaning.

Theorem 2.3 lists some basic properties of diagonally dominant matrices that are often used in this work.

THEOREM 2.3. [11, Theorem 1] *If $A \in \mathbb{R}^{n\times n}$ is row diagonally dominant, then*

(a) *Every principal submatrix of A is row diagonally dominant;*

(b) *PAP^T is row diagonally dominant for any permutation matrix $P \in \mathbb{R}^{n\times n}$;*

(c) *If $a_{11} \neq 0$ then the Schur complement of a_{11} in A is row diagonally dominant;*

(d) *If $\det A \neq 0$ then $\det A$ has the same sign as the product $a_{11}a_{22}\cdots a_{nn}$; and*

(e) *$|\det A(i', i')| \geq |\det A(i', j')|$, for all $i = 1, \dots, n$ and for all $j \neq i$, where we recall that $A(i', j')$ denotes the submatrix of A formed by deleting row i and column j of A.*

Diagonally dominant matrices have several other nice properties that are useful. For instance, strictly diagonally dominant matrices (i.e., diagonally dominant matrices as defined in Definition 2.1 with strict inequalities) are nonsingular and Gaussian elimination can be performed on them without interchanging rows or columns. This implies that any strictly diagonally dominant matrix A has a unique LDU factorization. A general diagonally dominant matrix A may be rank deficient, and in this case A may not have an LDU factorization. However, applying any *diagonal pivoting strategy* (i.e., pivoting with simultaneous and equal row and column permutations) to A always leads to a matrix PAP^T that has LDU factorization, $PAP^T = LDU$, where P is the permutation matrix defined by the pivoting strategy. If A is rank deficient, then D has diagonal entries that are zero and L and U may not be unique, even when P is fixed. We then consider the following unique form of the LDU factorization.

DEFINITION 2.4. [11, Definition 1] *A row diagonally dominant matrix $A \in \mathbb{R}^{n\times n}$ with rank r is said to have* LDU *factorization if there exist a unit lower triangular matrix $L_{11} \in \mathbb{R}^{r\times r}$, a unit upper triangular matrix $U_{11} \in \mathbb{R}^{r\times r}$, and a nonsingular diagonal matrix $D_{11} \in \mathbb{R}^{r\times r}$ such that $A = LDU$ where*

$$L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & I_{n-r} \end{bmatrix}, \qquad D = \begin{bmatrix} D_{11} & 0 \\ 0 & 0_{n-r} \end{bmatrix}, \qquad U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & I_{n-r} \end{bmatrix}.$$

The nontrivial entries of the L, D, and U factors in Definition 2.4 can be expressed in terms of minors of A. This is a classical result that we recall in Theorem 2.5, since it is fundamental in section 3, where the new perturbation bounds for the LDU factorization are obtained via detailed perturbation properties of the minors of A.

THEOREM 2.5. [15, p. 35] *If $A \in \mathbb{R}^{n\times n}$ has rank r and has* LDU *factorization as in Definition 2.4, then this factorization is unique and the nontrivial entries of $L = [l_{ij}]$, $D = \operatorname{diag}[d_1, \dots, d_r, 0, \dots, 0]$, and $U = [u_{ij}]$ are given by*

$$l_{ij} = \frac{\det A([1:j-1, i], 1:j)}{\det A(1:j, 1:j)}, \quad i > j \text{ and } j = 1, \dots, r, \tag{2.2}$$

$$d_i = \frac{\det A(1:i, 1:i)}{\det A(1:i-1, 1:i-1)}, \quad i = 1, \dots, r, \tag{2.3}$$

$$u_{ij} = \frac{\det A(1:i, [1:i-1, j])}{\det A(1:i, 1:i)}, \quad i < j \text{ and } i = 1, \dots, r, \tag{2.4}$$

where we define $\det A(1:0,1:0) := 1$.

Next, we revise two particular diagonal pivoting strategies which are used to compute via Gaussian elimination LDU factorizations of a row diagonally dominant matrix A that are RRDs, i.e., with both factors L and U well conditioned. First, recall that any diagonal pivoting strategy applied on A leads to a factor U that is also row diagonally dominant and, therefore, well-conditioned [26]. At each stage of Gaussian elimination with *complete diagonal pivoting*, the same row and column are exchanged to place in the pivot position the diagonal entry with the largest absolute value of the corresponding Schur complement. Observe that for row diagonally dominant matrices, complete diagonal pivoting coincides with standard complete pivoting but, despite this fact, it does not lead to an LDU factorization that is guaranteed to be a RRD, although in practice, the computed factor L is almost always well-conditioned. The *column diagonal dominance pivoting strategy* announced in the Introduction is much less known than complete diagonal pivoting, but applied on A computes a factor L which is column diagonally dominant and, hence, is always well-conditioned [26]. Some additional notation is needed to introduce such pivoting strategy.

In general, consider applying Gaussian elimination with a diagonal pivoting strategy to a row diagonally dominant matrix $A \in \mathbb{R}^{n\times n}$ with nonnegative diagonal entries. We assume that A is arranged for that diagonal pivoting strategy, which means that the permutation defined by the pivoting is applied to A in advance. In each stage Gaussian elimination makes zero all the entries below the diagonal of a certain column. Define $A^{(1)} := A$ and define $A^{(k+1)} := [a_{ij}^{(k+1)}] \in \mathbb{R}^{n\times n}$ to be the matrix obtained after k stages of Gaussian elimination have been performed. So, all the entries below the diagonal in the first k columns of $A^{(k+1)}$ are equal to zero, $A^{(k+1)}(1:k,:) = A^{(k)}(1:k,:)$, and, in addition, the following identity presented in [15] can be easily proved from properties of determinants:

$$a_{ij}^{(k+1)} = \frac{\det A([1:k,i],[1:k,j])}{\det A(1:k,1:k)} \tag{2.5}$$

for $k+1 \leq i,j \leq n$ and $1 \leq k \leq \min\{r, n-1\}$, where $r = \operatorname{rank}(A)$. It follows from Theorem 2.3-(c) that $A^{(k+1)}$ is row diagonally dominant and that $A^{(k+1)}(k+1:n, k+1:n)$ is also row diagonally dominant, and from Theorem 2.3-(a)-(d) and (2.5) that $A^{(k+1)}$ has nonnegative diagonal entries. Thus, Gaussian elimination applied on A generates a sequence of row diagonally dominant matrices with nonnegative diagonal entries $A^{(k)} \in \mathbb{R}^{n\times n}$, $k = 1, 2, \dots, \min\{n, r+1\}$, such that $A^{(k)}(k:n, k:n)$ is also row diagonally dominant. Observe that this implies that there is at least one column of $A^{(k)}(k:n,k:n)$ which is column diagonally dominant, i.e., $a_{ii}^{(k)} - \sum_{j=k, j\neq i}^{n} |a_{ji}^{(k)}| \geq 0$ for some $i = k, \dots, n$. Then, the *column diagonal dominance pivoting strategy* arranges A in such a way that

$$a_{kk}^{(k)} = \max_{k\leq i\leq n} \left\{ a_{ii}^{(k)} : a_{ii}^{(k)} - \sum_{j=k, j\neq i}^{n} |a_{ji}^{(k)}| \geq 0 \right\}, \tag{2.6}$$

for $k = 1, \dots, r$. This pivoting strategy was suggested in [26] for matrices with other structures and used for the first time in [33] for general row diagonally dominant matrices. It is immediate to see that column diagonal dominance pivoting produces a column diagonally dominant factor L. From an algorithmic point of view column diagonal dominance pivoting is implemented by exchanging, before performing the

k-th stage of Gaussian elimination, the same row and column to place in the pivot position (k,k) the maximal diagonal entry which is column diagonally dominant to get (2.6). At the end, we obtain a row diagonally dominant factor U as usual and, in addition, a column diagonally dominant factor L. Hence, by [26], the condition numbers of L and U can be bounded as

$$\kappa_\infty(L) \le n^2, \quad \kappa_\infty(U) \le 2\,n, \quad \kappa_1(L) \le 2\,n, \quad \kappa_1(U) \le n^2\,. \tag{2.7}$$

So, the LDU factorization of A arranged with the column diagonal dominance pivoting strategy is always a RRD.

The proof of our main result hinges on Theorem 2.5 and several results for determinants and minors of row diagonally dominant matrices that were proved in [11]. For completeness, we present two of these results here.

LEMMA 2.6. [11, Lemma 1] *Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ be such that $v \ge 0$. Denote the algebraic cofactors of A by $C_{ij} := (-1)^{i+j} \det A(i',j')$, $1 \le i,j \le n$. Then*

$$\det A = v_i C_{ii} + \sum_{j\neq i} (|a_{ij}| C_{ii} + a_{ij} C_{ij}), \quad i = 1, \dots, n\,,$$

with $v_i C_{ii} \ge 0$ and $|a_{ij}| C_{ii} + a_{ij} C_{ij} \ge 0$ for $j \ne i$.

In [11], given a matrix $A \in \mathbb{R}^{n\times n}$, the following notation was introduced for simplicity

$$g_{pq}^{(k+1)} := \det A([1:k,p],[1:k,q]), \tag{2.8}$$

for $1 \le k \le n-1$ and $k+1 \le p,q \le n$. Note that all of the determinants appearing in Theorem 2.5 and equation (2.5) are particular cases of the determinants defined in (2.8). Lemma 2.7 establishes relationships that are utilized in section 3 to provide perturbation results for nonprincipal minors.

LEMMA 2.7. [11, Lemma 6] *Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ be such that $v \ge 0$. For $k = 1, \dots, n-2$, $p \ne q$, and $k+1 \le p, q \le n$, let G_{ij} be the algebraic cofactor of $A([1:k,p],[1:k,q])$ for the entry a_{ij}. Then the minors defined in (2.8) satisfy*

$$g_{pq}^{(k+1)} = a_{p1} G_{p1} + \cdots + a_{pk} G_{pk} + a_{pq} G_{pq}\,, \tag{2.9}$$

$$2 g_{pp}^{(k+1)} \ge |a_{p1} G_{p1}| + \cdots + |a_{pk} G_{pk}| + |a_{pq} G_{pq}|\,, \tag{2.10}$$

and, for $1 \le i \le k$,

$$g_{pq}^{(k+1)} = \left(v_i + \sum_{j \notin \{1,\dots,k,q\}} |a_{ij}| \right) G_{ii} + \sum_{j \in \{1,\dots,k,q\}\setminus\{i\}} (a_{ij} G_{ij} + |a_{ij}| G_{ii})\,, \tag{2.11}$$

$$2 g_{pp}^{(k+1)} \ge \left(v_i + \sum_{j \notin \{1,\dots,k,q\}} |a_{ij}| \right) |G_{ii}| + \sum_{j \in \{1,\dots,k,q\}\setminus\{i\}} |a_{ij} G_{ij} + |a_{ij}| G_{ii}|\,. \tag{2.12}$$

3. Structured perturbation bounds for LDU factorizations of diagonally dominant matrices. We have explained in the Introduction that classical perturbation bounds for the LU factorization as those presented in [2, 7, 13, 30] and [20, page 194] are not useful for proving that the algorithm presented in [33] computes an accurate LDU factorization of row diagonally dominant matrices. The reason is that these classical bounds do not take into account the diagonally dominant structure and, as a consequence, they depend on the condition numbers of the factors and may be very large.

For any row diagonally dominant matrix A, a stronger perturbation theory for its LDU factorization has been presented in [11] and has been successfully used to prove rigorously that the algorithm in [33] with complete diagonal pivoting computes an accurate LDU factorization of A. The key components of this theory are to use as parameters the diagonally dominant parts and the off-diagonal entries of A introduced in Definition 2.2 and equation (2.1), and to preserve the diagonally dominant structure for getting perturbation bounds which are independent of any condition number and are always tiny for tiny perturbations. For completeness, we state the main perturbation result in [11] as follows.

THEOREM 3.1. [11, Theorem 3] *Let* $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ *be such that* $v \geq 0$. *Suppose* A *has* LDU *factorization* $A = LDU$ *as in Definition 2.4 with* $L = [l_{ij}] \in \mathbb{R}^{n\times n}$, $D = \mathrm{diag}[d_1, \ldots, d_n] \in \mathbb{R}^{n\times n}$, *and* $U = [u_{ij}] \in \mathbb{R}^{n\times n}$. *Let* $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ *be such that*

$$|\tilde{v} - v| \leq \epsilon v \quad \text{and} \quad |\tilde{A}_D - A_D| \leq \epsilon |A_D|, \qquad \text{for some } 0 \leq \epsilon < \frac{1}{2n}. \tag{3.1}$$

Then $\tilde{A}$ *has LDU factorization* $\tilde{A} = \tilde{L}\tilde{D}\tilde{U}$ *with* $\tilde{L} = [\tilde{l}_{ij}]$, $\tilde{D} = \mathrm{diag}[\tilde{d}_1, \ldots, \tilde{d}_n]$, *and* $\tilde{U} = [\tilde{u}_{ij}]$ *and the factors* $\tilde{L}$, $\tilde{D}$, *and* $\tilde{U}$ *satisfy,*

(a) *for* $i = 1, \ldots, n$,

$$|\tilde{d}_i - d_i| \leq \frac{2n\epsilon}{1 - 2n\epsilon}\, d_i\,;$$

(b) *for* $1 \leq i < j \leq n$,

$$|\tilde{u}_{ij} - u_{ij}| \leq 3n\epsilon, \quad \text{and} \quad \frac{\|\tilde{U} - U\|_\infty}{\|U\|_\infty} \leq 3n^2\epsilon;$$

(c) *and, if* A *is arranged for complete diagonal pivoting, for* $n \geq i > j \geq 1$,

$$|\tilde{l}_{ij} - l_{ij}| \leq \frac{3n\epsilon}{1 - 2n\epsilon}, \quad \text{and} \quad \frac{\|\tilde{L} - L\|_\infty}{\|L\|_\infty} \leq \frac{3n^2\epsilon}{1 - 2n\epsilon}.$$

Observe that the assumptions $v \geq 0$ and $0 \leq \epsilon < 1/(2n) < 1$ in Theorem 3.1 imply $\tilde{v} \geq 0$, and, so, $\tilde{A}$ is row diagonally dominant with nonnegative diagonal entries. In plain words, this means that the perturbations of A considered in (3.1) preserve the diagonally dominant structure. It is interesting to remark that the original statements of parts (a) and (b) of Theorem 3.1 presented in [11] remain valid under the wider assumption $0 \leq \epsilon < 1$ at the cost of somewhat complicating the bounds.

The perturbation bounds for D and U in Theorem 3.1 hold in general, but the bound on L only holds if the matrix A is arranged for complete diagonal pivoting. This is a critical assumption for the proof of part (c) in Theorem 3.1. Indeed, [11] provides an example where the perturbation of L may be of order 1 for very small ϵ if the complete diagonal pivoting strategy is not used. As discussed in Sections 1 and 2, the complete diagonal pivoting strategy, although useful for almost all matrices, does not guarantee a well-conditioned factor L and, therefore, it does not compute an LDU factorization which is guaranteed to be a RRD. It is then essential to demonstrate that a rank-revealing LDU factorization such as the one produced by the column diagonal dominance pivoting strategy is stable under the structured perturbations considered in Theorem 3.1. The present paper proves precisely this, by showing that a normwise perturbation bound on L similar to that in Theorem 3.1(c) holds when column diagonal dominance pivoting is used. This is the main result in this paper and is stated in Theorem 3.2.

THEOREM 3.2. *Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ be such that $v \geq 0$. Suppose that A is arranged for column diagonal dominance pivoting and hence has* LDU *factorization $A = LDU$. Let $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ be such that*

$$|\tilde{v} - v| \leq \epsilon v \quad \textit{and} \quad |\tilde{A}_D - A_D| \leq \epsilon |A_D|, \quad \textit{for some } 0 \leq \epsilon < \frac{1}{12\,n+1}. \tag{3.2}$$

Then $\tilde{A}$ has LDU *factorization $\tilde{A} = \tilde{L}\tilde{D}\tilde{U}$, parts* (a) *and* (b) *of Theorem 3.1 hold, and, in addition,*

$$\|\tilde{L} - L\|_1 \leq \frac{2\,n\,(4n-1)\,\epsilon}{1-(12\,n+1)\,\epsilon}. \tag{3.3}$$

Observe that $\|L\|_1 \geq 1$ and, as a consequence, the absolute normwise bound in (3.3) immediately implies that the same bound holds for the relative variation $\|\tilde{L} - L\|_1/\|L\|_1$. The assumption $0 \leq \epsilon < 1/(12\,n+1)$ in Theorem 3.2 implies that the bound in (3.3) is well-defined. The rest of this section is devoted to prove Theorem 3.2. Clearly, as a consequence of Theorem 3.1, we only need to prove (3.3), but this requires considerable efforts and the development of several auxiliary technical lemmas in advance. These lemmas may be also of interest for other purposes, and are presented in Section 3.1 together with the proof of Theorem 3.2.

3.1. Auxiliary lemmas and proof of Theorem 3.2. As part of the proof of Theorem 3.2, we need to consider auxiliary perturbations that are more general than those in (3.2). More precisely, *for a fixed p*, the p-th column of A will be perturbed in the particular way appearing in (3.5). In addition, in the first part of this section the perturbation parameter ϵ can be considered to satisfy $0 \leq \epsilon < 1$ and A can be any row diagonally dominant matrix. Only at the end of this section it will be imposed that A is arranged for column diagonal dominance pivoting. Next, we consider matrices $A = [a_{ij}] = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$, with $v \geq 0$, and $\tilde{A} = [\tilde{a}_{ij}] = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ that satisfy, for $0 \leq \epsilon < 1$,

$$|\tilde{v} - v| \leq \epsilon v, \tag{3.4}$$

$$|\tilde{a}_{ip} - a_{ip}| \leq \epsilon\,(v_i + |a_{ip}|), \quad \text{for } i \in \{1,\ldots,n\}\backslash\{p\}, \quad \text{and} \tag{3.5}$$

$$|\tilde{a}_{ij} - a_{ij}| \leq \epsilon\,|a_{ij}|, \quad \text{for } i \neq j,\ i \in \{1,\ldots,n\},\ j \in \{1,\ldots,n\}\backslash\{p\}. \tag{3.6}$$

This generalized perturbation can be equivalently expressed as

$$\tilde{v}_i = v_i(1+\phi_i), \quad \text{with } |\phi_i| \leq \epsilon, \text{ for } i \in \{1:n\}, \tag{3.7}$$

$$\tilde{a}_{ip} = a_{ip}(1+\phi'_{ip}) + \phi_{ip}\,v_i, \text{ with } |\phi_{ip}| \leq \epsilon,\ \phi'_{ip} = \phi_{ip}\,\text{sign}(a_{ip}), \text{ for } i \in \{1:n\}\backslash\{p\}, \tag{3.8}$$

$$\tilde{a}_{ij} = a_{ij}(1+\phi_{ij}), \text{ with } |\phi_{ij}| \leq \epsilon, \text{ for } i \neq j,\ i \in \{1:n\},\ j \in \{1:n\}\backslash\{p\}. \tag{3.9}$$

Observe that from (3.4), we obtain again that $\tilde{v} \geq 0$ holds and, so, this generalized perturbation also preserves the row diagonally dominant structure, as well as the nonnegativity of the diagonal entries.

Our first lemma generalizes [11, Lemma 3] to include the generalized perturbation defined in (3.4)-(3.5)-(3.6).

LEMMA 3.3. *Let* $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ *be such that* $v \geq 0$ *and let* $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ *satisfy* (3.4)-(3.5)-(3.6) *with* $0 \leq \epsilon < 1$. *Suppose* $\tilde{A}^{[i]} = \mathcal{D}(\tilde{A}_D^{[i]}, \tilde{v}^{[i]}) \in \mathbb{R}^{n\times n}$ *is a matrix that differs from* A *in only the* i*th row and whose* i*th row is the same as the* i*th row of* $\tilde{A}$. *Then*

$$\det \tilde{A}^{[i]} = (\det A)(1 + \eta_i), \quad \textit{where } |\eta_i| \leq 3\,\epsilon. \tag{3.10}$$

Proof. Let $A = [a_{jk}]$, $v = [v_j]$, $\tilde{A} = [\tilde{a}_{jk}]$, and $\tilde{v} = [\tilde{v}_j]$. We consider the cofactor expansion of $\det \tilde{A}^{[i]}$ across row i. Let $\tilde{C}_{ij}$ be the algebraic cofactor of $\tilde{A}^{[i]}$ corresponding to $\tilde{a}_{ij}^{[i]}$ and C_{ij} be the algebraic cofactor of A corresponding to a_{ij}. Then, $\tilde{C}_{ij} = C_{ij}$. We need to discuss two cases separately.

Case 1: $i = p$. By Lemma 2.6, we have

$$\det \tilde{A}^{[p]} = \tilde{v}_p \tilde{C}_{pp} + \sum_{j\neq p} (|\tilde{a}_{pj}|\tilde{C}_{pp} + \tilde{a}_{pj}\tilde{C}_{pj}) = \tilde{v}_p C_{pp} + \sum_{j\neq p} (|\tilde{a}_{pj}|C_{pp} + \tilde{a}_{pj}C_{pj})$$

and a similar equation for $\det A$. Lemma 2.6 further implies that $v_p C_{pp} \geq 0$ and $|a_{pj}|C_{pp} + a_{pj}C_{pj} \geq 0$. Using (3.7) and (3.9), we obtain

$$\det \tilde{A}^{[p]} = \det A + \phi_p v_p C_{pp} + \sum_{j\neq p} \phi_{pj}(|a_{pj}|C_{pp} + a_{pj}C_{pj}).$$

It follows that

$$\begin{aligned}
|\det \tilde{A}^{[p]} - \det A| &\leq |\phi_p| v_p C_{pp} + \sum_{j\neq p} |\phi_{pj}|(|a_{pj}|C_{pp} + a_{pj}C_{pj}) \\
&\leq \epsilon v_p C_{pp} + \sum_{j\neq p} \epsilon(|a_{pj}|C_{pp} + a_{pj}C_{pj}) \\
&= \epsilon \det A,
\end{aligned}$$

which proves the lemma for this case.

Case 2: $i \neq p$. We again use Lemma 2.6, (3.7), (3.8), and (3.9) to obtain

$$\begin{aligned}
\det \tilde{A}^{[i]} &= \tilde{v}_i \tilde{C}_{ii} + \sum_{j\neq i} (|\tilde{a}_{ij}|\tilde{C}_{ii} + \tilde{a}_{ij}\tilde{C}_{ij}) \\
&= \tilde{v}_i C_{ii} + |\tilde{a}_{ip}|C_{ii} + \tilde{a}_{ip}C_{ip} + \sum_{j\neq i,p} (|\tilde{a}_{ij}|C_{ii} + \tilde{a}_{ij}C_{ij}) \\
&= v_i C_{ii} + v_i\phi_i C_{ii} + |\tilde{a}_{ip}|C_{ii} + a_{ip}C_{ip} + a_{ip}\phi'_{ip}C_{ip} + \phi_{ip}v_i C_{ip} \\
&\quad + \sum_{j\neq i,p} (|a_{ij}|C_{ii} + a_{ij}C_{ij}) + \sum_{j\neq i,p} \phi_{ij}\left(|a_{ij}|C_{ii} + a_{ij}C_{ij}\right).
\end{aligned} \tag{3.11}$$

¿From (3.8), we have

$$|a_{ip}|(1 + \phi'_{ip}) - \epsilon v_i \leq |\tilde{a}_{ip}| \leq |a_{ip}|(1 + \phi'_{ip}) + \epsilon v_i\,, \tag{3.12}$$

and hence, from (3.11), Lemma 2.6, and $|C_{ip}| \leq C_{ii}$ (see Theorem 2.3(e)),

$$\begin{aligned}
\det \tilde{A}^{[i]} &\geq v_i C_{ii} + v_i \phi_i C_{ii} + |a_{ip}| C_{ii} + |a_{ip}| \phi'_{ip} C_{ii} - \epsilon v_i C_{ii} + a_{ip} C_{ip} + a_{ip} \phi'_{ip} C_{ip} \\
&\quad + \phi_{ip} v_i C_{ip} + \sum_{j \neq i,p} (|a_{ij}| C_{ii} + a_{ij} C_{ij}) + \sum_{j \neq i,p} \phi_{ij} (|a_{ij}| C_{ii} + a_{ij} C_{ij}) \\
&= \det A + \phi_i v_i C_{ii} + \phi'_{ip} (|a_{ip}| C_{ii} + a_{ip} C_{ip}) - \epsilon v_i C_{ii} + \phi_{ip} v_i C_{ip} \\
&\quad + \sum_{j \neq i,p} \phi_{ij} (|a_{ij}| C_{ii} + a_{ij} C_{ij}) \\
&\geq \det A - \epsilon v_i C_{ii} - \epsilon (|a_{ip}| C_{ii} + a_{ip} C_{ip}) - \epsilon v_i C_{ii} - \epsilon v_i C_{ii} \\
&\quad - \epsilon \sum_{j \neq i,p} (|a_{ij}| C_{ii} + a_{ij} C_{ij}) \\
&= \det A - 3 \epsilon v_i C_{ii} - \epsilon \sum_{j \neq i} (|a_{ij}| C_{ii} + a_{ij} C_{ij}) \\
&\geq \det A - 3 \epsilon \det A \, .
\end{aligned}$$

Similarly,

$$\begin{aligned}
\det \tilde{A}^{[i]} &\leq v_i C_{ii} + v_i \phi_i C_{ii} + |a_{ip}| C_{ii} + |a_{ip}| \phi'_{ip} C_{ii} + \epsilon v_i C_{ii} + a_{ip} C_{ip} + a_{ip} \phi'_{ip} C_{ip} \\
&\quad + \phi_{ip} v_i C_{ip} + \sum_{j \neq i,p} (|a_{ij}| C_{ii} + a_{ij} C_{ij}) + \sum_{j \neq i,p} \phi_{ij} (|a_{ij}| C_{ii} + a_{ij} C_{ij}) \\
&\leq \det A + \epsilon v_i C_{ii} + \epsilon (|a_{ip}| C_{ii} + a_{ip} C_{ip}) + \epsilon v_i C_{ii} + \epsilon v_i |C_{ip}| \\
&\quad + \epsilon \sum_{j \neq i,p} (|a_{ij}| C_{ii} + a_{ij} C_{ij}) \\
&\leq \det A + 3 \epsilon v_i C_{ii} + \epsilon \sum_{j \neq i} (|a_{ij}| C_{ii} + a_{ij} C_{ij}) \\
&\leq \det A + 3 \epsilon \det A.
\end{aligned}$$

Thus,

$$|\det \tilde{A}^{[i]} - \det A| \leq 3 \epsilon \det A \, ,$$

which proves the lemma. □

Lemma 3.4 below uses Lemma 3.3 to present a similar perturbation bound for the principal minors of a row diagonally dominant matrix under the structured perturbations defined in (3.4)-(3.5)-(3.6). It generalizes [11, Lemma 4] to this type of perturbations.

Lemma 3.4. *Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n \times n}$ be such that $v \geq 0$ and let $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n \times n}$ satisfy* (3.4)-(3.5)-(3.6) *with $0 \leq \epsilon < 1/2$. Suppose $\tilde{A}^{[i]} = \mathcal{D}(\tilde{A}_D^{[i]}, \tilde{v}^{[i]}) \in \mathbb{R}^{n \times n}$ is a matrix that differs from A in only the ith row and whose ith row is the same as the ith row of $\tilde{A}$. Let $1 \leq i_1 < i_2 < \cdots < i_q \leq n$ and $\alpha = \{i_1, i_2, \ldots, i_q\}$, and recall that $A(\alpha, \alpha)$ denotes the principal submatrix of A that lies in rows and columns indexed by α. Then*

$$\det \tilde{A}^{[i]}(\alpha, \alpha) = \begin{cases} \det A(\alpha, \alpha) \, , & \text{if } i \notin \alpha \, , \\ (\det A(\alpha, \alpha))(1 + \delta_i^{(\alpha)}) \, , & \text{if } i \in \alpha \, , \end{cases} \tag{3.13}$$

where $|\delta_i^{(\alpha)}| \leq 6\epsilon$ if $p \notin \alpha$ and $|\delta_i^{(\alpha)}| \leq 3\epsilon$ if $p \in \alpha$. Furthermore,

$$\det \tilde{A}(\alpha, \alpha) = (\det A(\alpha, \alpha))(1 + \eta_1^{(\alpha)}) \cdots (1 + \eta_q^{(\alpha)}) \, , \tag{3.14}$$

where $|\eta_k^{(\alpha)}| \le 6\epsilon$ *if* $p \notin \alpha$ *and* $|\eta_k^{(\alpha)}| \le 3\epsilon$ *if* $p \in \alpha$, *for* $k = 1, \ldots, q$.

Proof. We prove first (3.13). Assume $i \in \alpha$, otherwise the result is trivial. Since $A = [a_{jk}]$ and $\tilde{A}^{[i]} = [\tilde{a}_{jk}^{[i]}]$ are row diagonally dominant with nonnegative diagonal entries, then so are $A(\alpha,\alpha)$ and $\tilde{A}^{[i]}(\alpha,\alpha)$. Hence, we can parameterize them in terms of their diagonally dominant parts and off-diagonal entries. Let

$$A(\alpha,\alpha) = \mathcal{D}(A_D(\alpha,\alpha), w) \quad \text{and} \quad \tilde{A}^{[i]}(\alpha,\alpha) = \mathcal{D}(\tilde{A}_D^{[i]}(\alpha,\alpha), \tilde{w}^{[i]}),$$

where $w = [w_j]$, $\tilde{w}^{[i]} = [\tilde{w}_j^{[i]}] \in \mathbb{R}^q$. For simplicity, the entries and diagonally dominant parts of $A(\alpha,\alpha)$ and $\tilde{A}^{[i]}(\alpha,\alpha)$ are indexed with the indices $i_1, i_2, \ldots, i_q$ in α. Let $v = [v_j]$ and $\tilde{v}^{[i]} = [\tilde{v}_j^{[i]}]$. Let us compare the diagonally dominant parts of $A(\alpha,\alpha)$ and $\tilde{A}^{[i]}(\alpha,\alpha)$. To this purpose observe that $w_j = \tilde{w}_j^{[i]}$ if $j \in \alpha \setminus \{i\}$,

$$w_i = a_{ii} - \sum_{j\in\alpha\setminus\{i\}} |a_{ij}| = \left(v_i + \sum_{j\neq i} |a_{ij}| \right) - \sum_{j\in\alpha\setminus\{i\}} |a_{ij}| = v_i + \sum_{j\notin\alpha} |a_{ij}|, \qquad (3.15)$$

and, similarly, $\tilde{w}_i^{[i]} = \tilde{v}_i^{[i]} + \sum_{j\notin\alpha} |\tilde{a}_{ij}^{[i]}|$. Thus, we have

$$|\tilde{w}_i^{[i]} - w_i| = \left| \tilde{v}_i^{[i]} - v_i + \sum_{j\notin\alpha} \left(|\tilde{a}_{ij}^{[i]}| - |a_{ij}| \right) \right| \le |\tilde{v}_i^{[i]} - v_i| + \sum_{j\notin\alpha} \left| \tilde{a}_{ij}^{[i]} - a_{ij} \right|. \qquad (3.16)$$

If $p \in \alpha$, then (3.4) and (3.6) imply

$$|\tilde{w}_i^{[i]} - w_i| \le \epsilon v_i + \sum_{j\notin\alpha} \epsilon |a_{ij}| = \epsilon \left(v_i + \sum_{j\notin\alpha} |a_{ij}| \right) = \epsilon w_i.$$

Since $w_i \ge v_i$, by (3.15), the off-diagonal entries of $\tilde{A}^{[i]}(\alpha,\alpha)$ and $A(\alpha,\alpha)$ satisfy conditions (3.5)-(3.6) for their parameters. Therefore, we can apply Lemma 3.3 to $\tilde{A}^{[i]}(\alpha,\alpha)$ and $A(\alpha,\alpha)$ to obtain that, if $p \in \alpha$, then

$$\det \tilde{A}^{[i]}(\alpha,\alpha) = (\det A(\alpha,\alpha))(1 + \delta_i^{(\alpha)})$$

with $|\delta_i^{(\alpha)}| \le 3\epsilon$. If $p \notin \alpha$, then, from (3.16) and (3.4)-(3.5)-(3.6), we get

$$\begin{aligned} |\tilde{w}_i^{[i]} - w_i| &\le |\tilde{v}_i^{[i]} - v_i| + \sum_{j\notin\alpha, j\neq p} \left| \tilde{a}_{ij}^{[i]} - a_{ij} \right| + |\tilde{a}_{ip}^{[i]} - a_{ip}| \\ &\le \epsilon v_i + \sum_{j\notin\alpha, j\neq p} \epsilon |a_{ij}| + \epsilon(v_i + |a_{ip}|) \\ &= 2\epsilon v_i + \sum_{j\notin\alpha} \epsilon |a_{ij}| \le 2\epsilon w_i. \end{aligned}$$

Again, the off-diagonal entries of $\tilde{A}^{[i]}(\alpha,\alpha)$ and $A(\alpha,\alpha)$ satisfy (3.5)-(3.6) for their parameters. So, we can apply Lemma 3.3 to $\tilde{A}^{[i]}(\alpha,\alpha)$ and $A(\alpha,\alpha)$, but this time with ϵ replaced by 2ϵ, which requires $2\epsilon < 1$, to obtain

$$\det \tilde{A}^{[i]}(\alpha,\alpha) = (\det A(\alpha,\alpha))(1 + \delta_i^{(\alpha)})$$

with $|\delta_i^{(\alpha)}| \leq 6\epsilon$, for $p \notin \alpha$. This proves (3.13).

Finally, we prove (3.14). To this purpose, consider that the perturbed submatrix $\tilde{A}(\alpha,\alpha)$ can be obtained from $A(\alpha,\alpha)$ by a sequence of "only one row" at a time perturbations. By (3.13), each of these "only one row" perturbations produces a determinant that is equal to the determinant before the perturbation times a factor $1+\eta$, with $|\eta| \leq 6\epsilon$ if $p \notin \alpha$ and $|\eta| \leq 3\epsilon$ if $p \in \alpha$. ◻

Lemma 3.5 considers the variation of certain nonprincipal minors of a row diagonally dominant matrix A under the structured perturbations defined in (3.4)-(3.5)-(3.6). Observe that the last row of these minors corresponds precisely to the fixed index p appearing in (3.4)-(3.5)-(3.6). In Lemma 3.5, the minors of A that are of interest are denoted as in (2.8), that is,

$$g_{pq}^{(k+1)} = \det A([1:k,p],[1:k,q]), \tag{3.17}$$

for $1 \leq k \leq n-1$ and $k+1 \leq p,q \leq n$. We also denote by $\left(\tilde{g}^{[i]}\right)_{pq}^{(k+1)}$ and $\tilde{g}_{pq}^{(k+1)}$ the corresponding minors of the perturbed matrices $\tilde{A}^{[i]}$ and $\tilde{A}$, respectively, which were defined in Lemmas 3.3 and 3.4.

Lemma 3.5. *Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ be such that $v \geq 0$ and let $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ satisfy (3.4)-(3.5)-(3.6) with $0 \leq \epsilon < 1/2$. Suppose $\tilde{A}^{[i]} = \mathcal{D}(\tilde{A}_D^{[i]}, \tilde{v}^{[i]}) \in \mathbb{R}^{n\times n}$ is a matrix that differs from A in only the ith row and whose ith row is the same as the ith row of $\tilde{A}$. Let $1 \leq k \leq n-2$, $k+1 \leq p,q \leq n$, and $p \neq q$, where p is the fixed index in (3.4)-(3.5)-(3.6). Then, the following statements hold for the minors in (3.17):*

(a) $\left|\left(\tilde{g}^{[i]}\right)_{pq}^{(k+1)} - g_{pq}^{(k+1)}\right| \leq \begin{cases} 0, & \text{if } i \notin \{1:k,p\} \\ 4\epsilon g_{pp}^{(k+1)}, & \text{if } i \in \{1:k,p\} \end{cases}$;

(b) $\left|\tilde{g}_{pq}^{(k+1)} - g_{pq}^{(k+1)}\right| \leq \frac{4}{3}\left((1+3\epsilon)^{k+1} - 1\right) g_{pp}^{(k+1)}$.

Proof. Let $A = [a_{jk}]$, $\tilde{A}^{[i]} = [\tilde{a}_{jk}^{[i]}]$, $\tilde{A} = [\tilde{a}_{jk}]$, $v = [v_j]$, $\tilde{v}^{[i]} = [\tilde{v}_j^{[i]}]$, and $\tilde{v} = [\tilde{v}_j]$. We prove first part (a) for $i \in \{1:k,p\}$, as the case $i \notin \{1:k,p\}$ is trivial. For $j \in \{1:k,q\}$, let G_{ij} be the algebraic cofactor of $A([1:k,p],[1:k,q])$ for the entry a_{ij}, and note that this is also the algebraic cofactor of $\tilde{A}^{[i]}([1:k,p],[1:k,q])$ for the entry $\tilde{a}_{ij}^{[i]}$.

If $1 \leq i \leq k$, applying Lemma 2.7 yields

$$\begin{aligned}\left(\tilde{g}^{[i]}\right)_{pq}^{(k+1)} &= \left(\tilde{v}_i^{[i]} + \sum_{j\notin\{1:k,q\}} |\tilde{a}_{ij}^{[i]}|\right) G_{ii} + \sum_{j\in\{1:k,q\}\setminus\{i\}} \left(\tilde{a}_{ij}^{[i]} G_{ij} + |\tilde{a}_{ij}^{[i]}| G_{ii}\right) \\ &= \left(\tilde{v}_i + \sum_{j\notin\{1:k,p,q\}} |\tilde{a}_{ij}|\right) G_{ii} + |\tilde{a}_{ip}| G_{ii} + \sum_{j\in\{1:k,q\}\setminus\{i\}} \left(\tilde{a}_{ij} G_{ij} + |\tilde{a}_{ij}| G_{ii}\right).\end{aligned}$$

Similarly, we have

$$g_{pq}^{(k+1)} = \left(v_i + \sum_{j\notin\{1:k,p,q\}} |a_{ij}|\right) G_{ii} + |a_{ip}| G_{ii} + \sum_{j\in\{1:k,q\}\setminus\{i\}} \left(a_{ij} G_{ij} + |a_{ij}| G_{ii}\right).$$

Using (3.7) and (3.9), we can write

$$\left(\tilde{g}^{[i]}\right)_{pq}^{(k+1)} = g_{pq}^{(k+1)} + \left(v_i\phi_i + \sum_{j\notin\{1:k,p,q\}} \phi_{ij}|a_{ij}|\right) G_{ii} + \sum_{j\in\{1:k,q\}\setminus\{i\}} \phi_{ij}\left(a_{ij}G_{ij} + |a_{ij}|G_{ii}\right) + (|\tilde{a}_{ip}| - |a_{ip}|)G_{ii}\,. \tag{3.18}$$

Therefore, using (3.5), we get

$$\begin{aligned}\left|\left(\tilde{g}^{[i]}\right)_{pq}^{(k+1)} - g_{pq}^{(k+1)}\right| &\le \epsilon\Bigg[\left(v_i + \sum_{j\notin\{1:k,p,q\}} |a_{ij}|\right)|G_{ii}| \\ &\qquad + \sum_{j\in\{1:k,q\}\setminus\{i\}} |a_{ij}G_{ij} + |a_{ij}|G_{ii}| + |a_{ip}||G_{ii}| + v_i|G_{ii}|\Bigg] \\ &\le 2\epsilon\left[\left(v_i + \sum_{j\notin\{1:k,q\}} |a_{ij}|\right)|G_{ii}| + \sum_{j\in\{1:k,q\}\setminus\{i\}} |a_{ij}G_{ij} + |a_{ij}|G_{ii}|\right] \\ &\le 4\epsilon g_{pp}^{(k+1)}\,,\end{aligned}$$

where Lemma 2.7 has been used in the last inequality. This proves part (a) for $1 \le i \le k$.

Now, for $i = p$, use Lemma 2.7 and (3.9) to obtain

$$\begin{aligned}\left(\tilde{g}^{[p]}\right)_{pq}^{(k+1)} &= \sum_{j\in\{1:k,q\}} \tilde{a}_{pj}^{[p]}G_{pj} = \sum_{j\in\{1:k,q\}} a_{pj}(1+\phi_{pj})G_{pj} \\ &= \sum_{j\in\{1:k,q\}} a_{pj}G_{pj} + \sum_{j\in\{1:k,q\}} \phi_{pj}a_{pj}G_{pj} = g_{pq}^{(k+1)} + \sum_{j\in\{1:k,q\}} \phi_{pj}a_{pj}G_{pj}\,.\end{aligned}$$

Thus,

$$\left|\left(\tilde{g}^{[p]}\right)_{pq}^{(k+1)} - g_{pq}^{(k+1)}\right| \le \epsilon \sum_{j\in\{1:k,q\}} |a_{pj}G_{pj}| \le \epsilon(2g_{pp}^{(k+1)}), \tag{3.19}$$

where Lemma 2.7 has been used again in the last inequality. This proves part (a) for $i = p$ and completes the whole proof of this part.

For part (b), consider obtaining $\tilde{A}$ from A by a sequence of only one row at a time perturbations. Note that each matrix in this sequence is row diagonally dominant with nonnegative diagonals. The variation in $g_{pq}^{(k+1)}$ is a consequence only of the perturbations of rows with indices in $\{1:k,p\}$. Let α be a subset of $\{1:k,p\}$ and denote by $(\tilde{g}^{\alpha})_{pq}^{(k+1)}$ the minor corresponding to a matrix obtained from A through perturbations in the rows with indices in α only. Thus

$$\begin{aligned}\left|\tilde{g}_{pq}^{(k+1)} - g_{pq}^{(k+1)}\right| &= \left|\left(\tilde{g}^{\{1:k,p\}}\right)_{pq}^{(k+1)} - g_{pq}^{(k+1)}\right| \\ &\le \left|\left(\tilde{g}^{\{1:k,p\}}\right)_{pq}^{(k+1)} - \left(\tilde{g}^{\{1:k\}}\right)_{pq}^{(k+1)}\right| + \cdots + \left|\left(\tilde{g}^{\{1\}}\right)_{pq}^{(k+1)} - g_{pq}^{(k+1)}\right|.\end{aligned}$$

Apply part (a) to each term in this sum to obtain,

$$\left|\tilde{g}_{pq}^{(k+1)} - g_{pq}^{(k+1)}\right| \le 4\epsilon \left[\left(\tilde{g}^{\{1:k\}}\right)_{pp}^{(k+1)} + \cdots + g_{pp}^{(k+1)}\right]$$

and then apply Lemma 3.4 to each term in the sum above to get

$$\left|\tilde{g}_{pq}^{(k+1)} - g_{pq}^{(k+1)}\right| \le 4\epsilon \left[(1+3\epsilon)^k + \cdots + 1\right] g_{pp}^{(k+1)} = \frac{4}{3}\left((1+3\epsilon)^{k+1} - 1\right) g_{pp}^{(k+1)}.$$

□

Lemma 3.6 below is the key lemma towards the proof of our main result, i.e., Theorem 3.2. Lemma 3.6 considers row diagonally dominant matrices A and $\tilde{A}$ satisfying the standard perturbation (3.2) and constructs from them, via some elementary column operations, two new row diagonally dominant matrices with nonnegative diagonals B and $\tilde{B} \in \mathbb{R}^{(k+2)\times(k+2)}$. Furthermore, Lemma 3.6 proves, after considerable effort, that B and $\widetilde{B}$ satisfy the generalized perturbation (3.4)-(3.5)-(3.6). We warn the reader that the proof of Theorem 3.2 relies in applying Lemma 3.5 to the matrices B and $\tilde{B}$. As B is constructed from A, we label the $k+2$ rows and columns of B using the indices $\{1:k,p,q\}$. While not traditional, this labeling is useful because we can easily compare entries in B to entries in A. Thus, the $(k+1)$th row and column of B correspond to the pth row and column of A and similarly the $(k+2)$nd row and column of B correspond to the qth row and column of A. Note also that in Lemma 3.6 the condition $\epsilon < 1/5$ is imposed with the only purpose of guaranteeing $\delta < 1/2$ in (3.23)-(3.24)-(3.25), which is necessary to apply Lemma 3.5 to B and $\tilde{B}$.

Lemma 3.6. *Let $A = [a_{ij}] = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ be such that $v \ge 0$ and let $\tilde{A} = [\tilde{a}_{ij}] = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ be a matrix that satisfies*

$$|\tilde{v} - v| \le \epsilon v \quad \text{and} \quad |\tilde{A}_D - A_D| \le \epsilon |A_D|, \quad \text{for some } 0 \le \epsilon < \frac{1}{5}. \tag{3.20}$$

Let $A^{(k+1)} = [a_{ij}^{(k+1)}] \in \mathbb{R}^{n\times n}$ be the matrix obtained after k stages of Gaussian elimination have been performed on A and, for a fixed index p such that $k+1 \le p \le n$, let $s_j = \operatorname{sign}(a_{pj}^{(k+1)})$ for $j = 1,\ldots,n$. Let $1 \le k \le n-2$ and let $B = [b_{ij}] \in \mathbb{R}^{(k+2)\times(k+2)}$ be a matrix whose rows and columns are indexed by $i,j \in \{1,2,\cdots,k,p,q\}$ and is defined as follows

$$\begin{cases} b_{ij} = a_{ij}\,, & \text{for } i \in \{1:k,p,q\} \text{ and } j \in \{1:k,q\} \\ b_{ip} = a_{ip} - \displaystyle\sum_{j\notin\{1:k,p,q\}} s_j a_{ij}\,, & \text{for } i \in \{1:k,p,q\} \end{cases}. \tag{3.21}$$

Similarly, let $\tilde{B} = [\tilde{b}_{ij}] \in \mathbb{R}^{(k+2)\times(k+2)}$ be defined by

$$\begin{cases} \tilde{b}_{ij} = \tilde{a}_{ij}\,, & \text{for } i \in \{1:k,p,q\} \text{ and } j \in \{1:k,q\} \\ \tilde{b}_{ip} = \tilde{a}_{ip} - \displaystyle\sum_{j\notin\{1:k,p,q\}} s_j \tilde{a}_{ij}\,, & \text{for } i \in \{1:k,p,q\} \end{cases}. \tag{3.22}$$

Then B and $\tilde{B}$ are row diagonally dominant matrices with nonnegative diagonal entries. In addition, B and $\tilde{B}$ can be parameterized as $B = \mathcal{D}(B_D, w)$ and $\tilde{B} = \mathcal{D}(\tilde{B}_D, \tilde{w})$ and satisfy

$$|\tilde{w} - w| \le \delta w\,, \tag{3.23}$$

$$|\tilde{b}_{ip} - b_{ip}| \le \delta(w_i + |b_{ip}|)\,, \quad \text{for } i \in \{1:k,q\}, \quad \text{and }, \tag{3.24}$$

$$|\tilde{b}_{ij} - b_{ij}| \le \delta\,|b_{ij}|\,, \quad \text{for } i \ne j,\ i \in \{1:k,p,q\},\ j \in \{1:k,q\}, \tag{3.25}$$

where $\delta = \dfrac{2\epsilon}{1-\epsilon}$.

Proof. Using that A is row diagonally dominant and has nonnegative diagonal entries, we have for $i \in \{1:k,q\}$

$$\sum_{j\in\{1:k,p,q\}\setminus\{i\}} |b_{ij}| = \sum_{j\in\{1:k,q\}\setminus\{i\}} |b_{ij}| + |b_{ip}| = \sum_{j\in\{1:k,q\}\setminus\{i\}} |a_{ij}| + \left| a_{ip} - \sum_{j\notin\{1:k,p,q\}} s_j a_{ij} \right|$$
$$\leq \sum_{j\in\{1:k,q\}\setminus\{i\}} |a_{ij}| + |a_{ip}| + \sum_{j\notin\{1:k,p,q\}} |a_{ij}| = \sum_{j\neq i} |a_{ij}| \leq a_{ii} = b_{ii}\,,$$

and, for $i = p$,

$$\sum_{j\in\{1:k,q\}} |b_{pj}| = \sum_{j\in\{1:k,q\}} |a_{pj}| = \sum_{j\neq p} |a_{pj}| - \sum_{j\notin\{1:k,p,q\}} |a_{pj}|$$
$$\leq \sum_{j\neq p} |a_{pj}| - \sum_{j\notin\{1:k,p,q\}} s_j a_{pj} \leq a_{pp} - \sum_{j\notin\{1:k,p,q\}} s_j a_{pj} = b_{pp}\,.$$

Hence, B is row diagonally dominant with nonnegative diagonals. Using the same argument, the row diagonal dominance of $\tilde{A}$, and $\tilde{a}_{ii} \geq 0$, we can show that $\tilde{B}$ is row diagonally dominant with nonnegative diagonals as well. Thus, we can parameterize B and $\tilde{B}$ in terms of their diagonally dominant parts and off-diagonal entries. Let $B = \mathcal{D}(B_D, w)$ and $\tilde{B} = \mathcal{D}(\tilde{B}_D, \tilde{w})$ with $w = [w_i]$ and $\tilde{w} = [\tilde{w}_i] \in \mathbb{R}^{k+2}$. Now, note that, for $i \in \{1:k,p,q\}$, $j \in \{1:k,q\}$, and $i \neq j$, we have

$$|\tilde{b}_{ij} - b_{ij}| = |\tilde{a}_{ij} - a_{ij}| \leq \epsilon |a_{ij}| = \epsilon |b_{ij}|,$$

and, for $i \in \{1:k,q\}$ and $j = p$, we use $|a_{ip}| \leq |a_{ip}| + v_i$ to get

$$|\tilde{b}_{ip} - b_{ip}| = \left| \left(\tilde{a}_{ip} - \sum_{j\notin\{1:k,p,q\}} s_j \tilde{a}_{ij} \right) - \left(a_{ip} - \sum_{j\notin\{1:k,p,q\}} s_j a_{ij} \right) \right|$$
$$\leq |\tilde{a}_{ip} - a_{ip}| + \sum_{j\notin\{1:k,p,q\}} |\tilde{a}_{ij} - a_{ij}| \leq \epsilon\left(|a_{ip}| + v_i\right) + \epsilon \sum_{j\notin\{1:k,p,q\}} |a_{ij}|$$
$$= \epsilon \left(\sum_{j\notin\{1:k,q\}} |a_{ij}| + v_i \right) = \epsilon \left(\sum_{j\notin\{1:k,q\}} |a_{ij}| + a_{ii} - \sum_{j\neq i} |a_{ij}| \right)$$
$$= \epsilon \left(a_{ii} - \sum_{j\in\{1:k,q\}\setminus\{i\}} |a_{ij}| \right) = \epsilon \left(b_{ii} - \sum_{j\in\{1:k,q\}\setminus\{i\}} |b_{ij}| \right)$$
$$= \epsilon \left(b_{ii} - \sum_{j\in\{1:k,p,q\}\setminus\{i\}} |b_{ij}| + |b_{ip}| \right) = \epsilon(w_i + |b_{ip}|).$$

Thus, we have proved (3.24)-(3.25) for the off-diagonal entries of B and $\tilde{B}$. Now we focus on the diagonally dominant parts. Let $i \in \{1:k,q\}$ and observe

$$\begin{aligned}
w_i &= b_{ii} - \sum_{j\in\{1:k,p,q\}\setminus\{i\}} |b_{ij}| = b_{ii} - \sum_{j\in\{1:k,q\}\setminus\{i\}} |b_{ij}| - |b_{ip}| \\
&= a_{ii} - \sum_{j\in\{1:k,q\}\setminus\{i\}} |a_{ij}| - \left| a_{ip} - \sum_{j\notin\{1:k,p,q\}} s_j a_{ij} \right| \\
&= v_i + \sum_{j\notin\{1:k,q\}} |a_{ij}| - \left| a_{ip} - \sum_{j\notin\{1:k,p,q\}} s_j a_{ij} \right| \\
&= v_i + \sum_{j\notin\{1:k,p,q\}} |a_{ij}| + |a_{ip}| - \left| a_{ip} - \sum_{j\notin\{1:k,p,q\}} s_j a_{ij} \right| . \qquad (3.26)
\end{aligned}$$

Similarly, we have

$$\tilde{w}_i = \tilde{v}_i + \sum_{j\notin\{1:k,p,q\}} |\tilde{a}_{ij}| + |\tilde{a}_{ip}| - \left| \tilde{a}_{ip} - \sum_{j\notin\{1:k,p,q\}} s_j \tilde{a}_{ij} \right| . \qquad (3.27)$$

Next, we will consider two cases.

Case 1: $\operatorname{sign}\left(a_{ip} - \sum_{j\notin\{1:k,p,q\}} s_j a_{ij}\right) = \operatorname{sign}\left(\tilde{a}_{ip} - \sum_{j\notin\{1:k,p,q\}} s_j \tilde{a}_{ij}\right) =: \theta$. Then,

$$\begin{aligned}
\tilde{w}_i &= \tilde{v}_i + \sum_{j\notin\{1:k,p,q\}} |\tilde{a}_{ij}| + |\tilde{a}_{ip}| - \theta \left(\tilde{a}_{ip} - \sum_{j\notin\{1:k,p,q\}} s_j \tilde{a}_{ij} \right) \\
&= \tilde{v}_i + \sum_{j\notin\{1:k,p,q\}} |\tilde{a}_{ij}| + |\tilde{a}_{ip}| - \theta \tilde{a}_{ip} + \theta \sum_{j\notin\{1:k,p,q\}} s_j \tilde{a}_{ij} \\
&= \tilde{v}_i + \sum_{j\notin\{1:k,p,q\}} |\tilde{a}_{ij}|(1 + \theta s_j \operatorname{sign}(\tilde{a}_{ij})) + |\tilde{a}_{ip}|(1 - \theta \operatorname{sign}(\tilde{a}_{ip})) \\
&= \tilde{v}_i + \sum_{j\notin\{1:k,p,q\}} |\tilde{a}_{ij}|(1 + \theta s_j \operatorname{sign}(a_{ij})) + |\tilde{a}_{ip}|(1 - \theta \operatorname{sign}(a_{ip})),
\end{aligned}$$

where we have used that $\operatorname{sign}(\tilde{a}_{ij}) = \operatorname{sign}(a_{ij})$ for all $j \neq i$. Similarly, we have

$$w_i = v_i + \sum_{j\notin\{1:k,p,q\}} |a_{ij}|(1 + \theta s_j \operatorname{sign}(a_{ij})) + |a_{ip}|(1 - \theta \operatorname{sign}(a_{ip}))$$

and, hence,

$$\begin{aligned}
|\tilde{w}_i - w_i| &\le |\tilde{v}_i - v_i| + \sum_{j\notin\{1:k,p,q\}} \left|\tilde{a}_{ij} - a_{ij}\right|(1 + \theta s_j \operatorname{sign}(a_{ij})) \\
&\quad + \left|\tilde{a}_{ip} - a_{ip}\right|(1 - \theta \operatorname{sign}(a_{ip})) \\
&\le \epsilon v_i + \epsilon \sum_{j\notin\{1:k,p,q\}} |a_{ij}|(1 + \theta s_j \operatorname{sign}(a_{ij})) + \epsilon |a_{ip}|(1 - \theta \operatorname{sign}(a_{ip})) \\
&\le \epsilon w_i . \qquad (3.28)
\end{aligned}$$

Case 2: sign $\left(a_{ip} - \sum_{j \notin \{1:k,p,q\}} s_j a_{ij}\right) \neq$ sign $\left(\tilde{a}_{ip} - \sum_{j \notin \{1:k,p,q\}} s_j \tilde{a}_{ij}\right)$. In this case,

$$\begin{aligned}
&\left|\tilde{a}_{ip} - \sum_{j \notin \{1:k,p,q\}} s_j \tilde{a}_{ij}\right| + \left|a_{ip} - \sum_{j \notin \{1:k,p,q\}} s_j a_{ij}\right| \\
&\qquad = \left|\left(\tilde{a}_{ip} - \sum_{j \notin \{1:k,p,q\}} s_j \tilde{a}_{ij}\right) - \left(a_{ip} - \sum_{j \notin \{1:k,p,q\}} s_j a_{ij}\right)\right| \\
&\qquad \leq |\tilde{a}_{ip} - a_{ip}| + \sum_{j \notin \{1:k,p,q\}} |\tilde{a}_{ij} - a_{ij}| \\
&\qquad \leq \epsilon |a_{ip}| + \epsilon \sum_{j \notin \{1:k,p,q\}} |a_{ij}|, \qquad (3.29)
\end{aligned}$$

which, combined with (3.26) and (3.27), yields

$$\begin{aligned}
|\tilde{w}_i - w_i| &= \left|\left(\tilde{v}_i + \sum_{j \notin \{1:k,q\}} |\tilde{a}_{ij}| - \left|\tilde{a}_{ip} - \sum_{j \notin \{1:k,p,q\}} s_j \tilde{a}_{ij}\right|\right)\right. \\
&\qquad \left. - \left(v_i + \sum_{j \notin \{1:k,q\}} |a_{ij}| - \left|a_{ip} - \sum_{j \notin \{1:k,p,q\}} s_j a_{ij}\right|\right)\right| \\
&\leq |\tilde{v}_i - v_i| + \sum_{j \notin \{1:k,q\}} ||\tilde{a}_{ij}| - |a_{ij}|| \\
&\qquad + \left|\tilde{a}_{ip} - \sum_{j \notin \{1:k,p,q\}} s_j \tilde{a}_{ij}\right| + \left|a_{ip} - \sum_{j \notin \{1:k,p,q\}} s_j a_{ij}\right| \\
&\leq \epsilon v_i + \epsilon \sum_{j \notin \{1:k,q\}} |a_{ij}| + \epsilon \sum_{j \notin \{1:k,q\}} |a_{ij}| \\
&\leq 2\epsilon \left(v_i + \sum_{j \notin \{1:k,q\}} |a_{ij}|\right). \qquad (3.30)
\end{aligned}$$

So, from (3.26) and (3.29),

$$\begin{aligned}
w_i = v_i + \sum_{j \notin \{1:k,q\}} |a_{ij}| - \left|a_{ip} - \sum_{j \notin \{1:k,p,q\}} s_j a_{ij}\right| &\geq v_i + \sum_{j \notin \{1:k,q\}} |a_{ij}| - \epsilon \sum_{j \notin \{1:k,q\}} |a_{ij}| \\
\geq (1-\epsilon)\left(v_i + \sum_{j \notin \{1:k,q\}} |a_{ij}|\right).
\end{aligned}$$

Combining this inequality and (3.30), we have

$$|\tilde{w}_i - w_i| \leq \frac{2\epsilon}{1-\epsilon} w_i. \qquad (3.31)$$

The inequalities (3.28) and (3.31) prove the bound (3.23) for w_i with $i \in \{1:k,q\}$. Finally, we prove (3.23) for $i = p$. Note that

$$\begin{aligned}
w_p &= b_{pp} - \sum_{j\in\{1:k,q\}} |b_{pj}| = a_{pp} - \sum_{j\notin\{1:k,p,q\}} s_j a_{pj} - \sum_{j\in\{1:k,q\}} |a_{pj}| \\
&= v_p + \sum_{j\neq p} |a_{pj}| - \sum_{j\notin\{1:k,p,q\}} s_j a_{pj} - \sum_{j\in\{1:k,q\}} |a_{pj}| \\
&= v_p + \sum_{j\notin\{1:k,p,q\}} |a_{pj}| - \sum_{j\notin\{1:k,p,q\}} s_j a_{pj} = v_p + \sum_{j\notin\{1:k,p,q\}} (|a_{pj}| - s_j a_{pj}) \\
&= v_p + \sum_{j\notin\{1:k,p,q\}} |a_{pj}|(1 - s_j \mathrm{sign}(a_{pj}))\,.
\end{aligned}$$

Similarly, we have

$$\tilde{w}_p = \tilde{v}_p + \sum_{j\notin\{1:k,p,q\}} |\tilde{a}_{pj}|(1 - s_j \mathrm{sign}(\tilde{a}_{pj})) = \tilde{v}_p + \sum_{j\notin\{1:k,p,q\}} |\tilde{a}_{pj}|(1 - s_j \mathrm{sign}(a_{pj}))\,,$$

since $\mathrm{sign}(\tilde{a}_{pj}) = \mathrm{sign}(a_{pj})$. Thus,

$$\begin{aligned}
|\tilde{w}_p - w_p| &\leq |\tilde{v}_p - v_p| + \sum_{j\notin\{1:k,p,q\}} |\tilde{a}_{pj} - a_{pj}|(1 - s_j \mathrm{sign}(a_{pj})) \\
&\leq \epsilon v_p + \epsilon \sum_{j\notin\{1:k,p,q\}} |a_{pj}|(1 - s_j \mathrm{sign}(a_{pj})) = \epsilon w_p\,.
\end{aligned}$$

So, we have that $|\tilde{w}_i - w_i| \leq \dfrac{2\epsilon}{1-\epsilon} w_i$ for all $i \in \{1:k,p,q\}$. Lemma 3.6 is proved. ☐

The next lemma relates one of the minors of the matrix B defined in Lemma 3.6 with one minor of A. In the statement, we use the notation introduced in (2.8).

Lemma 3.7. *Let A and B be defined as in Lemma 3.6 and define*

$$(g_B)_{pp}^{(k+1)} := \det B([1:k,p],[1:k,p])\,.$$

Then, we have

(a) *If $g_{kk}^{(k)} \neq 0$, let $A^{(k+1)} = [a_{ij}^{(k+1)}]$ be the row diagonally dominant matrix with nonnegative diagonal entries obtained after k stages of Gaussian elimination have been performed on A, and let $A^{(k+1)}$ be parameterized as $A^{(k+1)} = \mathcal{D}(A_D^{(k+1)}, v^{(k+1)})$, with $v^{(k+1)} = [v_i^{(k+1)}]$. Then*

$$(g_B)_{pp}^{(k+1)} = \left(v_p^{(k+1)} + \left|a_{pq}^{(k+1)}\right|\right) g_{kk}^{(k)}\,.$$

(b) *If $g_{kk}^{(k)} = 0$, then $(g_B)_{pp}^{(k+1)} = 0$.*

Proof. Observe that $B([1:k,p],[1:k,p])$ and $A([1:k,p],[1:k,p])$ have columns 1 through k equal and for the last column, we have

$$B([1:k,p],p) = A([1:k,p],p) - \sum_{j\notin\{1:k,p,q\}} s_j\, A([1:k,p],j)\,.$$

Using the fact that the determinant is a linear function of any of its columns, assuming that the remaining columns are fixed, we obtain

$$(g_B)_{pp}^{(k+1)} = \det A([1:k,p],[1:k,p]) - \sum_{j\notin\{1:k,p,q\}} s_j \det A([1:k,p],[1:k,j])\,.$$

If $g_{kk}^{(k)} = \det A(1:k,1:k) \neq 0$, then

$$(g_B)_{pp}^{(k+1)} = g_{kk}^{(k)} \left(\frac{\det A([1:k,p],[1:k,p])}{\det A(1:k,1:k)} - \sum_{j\notin\{1:k,p,q\}} s_j \frac{\det A([1:k,p],[1:k,j])}{\det A(1:k,1:k)} \right).$$

By (2.5),

$$(g_B)_{pp}^{(k+1)} = g_{kk}^{(k)} \left(a_{pp}^{(k+1)} - \sum_{j\notin\{1:k,p,q\}} s_j a_{pj}^{(k+1)} \right) = g_{kk}^{(k)} \left(a_{pp}^{(k+1)} - \sum_{j\notin\{1:k,p,q\}} |a_{pj}^{(k+1)}| \right).$$

Since, $a_{pj}^{(k+1)} = 0$ for $1 \leq j \leq k$, we get

$$(g_B)_{pp}^{(k+1)} = g_{kk}^{(k)} \left(a_{pp}^{(k+1)} - \sum_{j\neq p} |a_{pj}^{(k+1)}| + |a_{pq}^{(k+1)}| \right) = g_{kk}^{(k)} \left(v_p^{(k+1)} + \left| a_{pq}^{(k+1)} \right| \right),$$

which proves part (a). Next, we prove part (b). If $g_{kk}^{(k)} = \det A(1:k,1:k) = 0$, then one of the pivots $a_{jj}^{(j)}$, for $1 \leq j \leq k$, in the Gaussian elimination for A must be 0. Since $A^{(j)}$ is still row diagonally dominant, then the jth row of $A^{(j)}$ must be entirely 0. Then applying $j-1$ stages of Gaussian elimination to the row diagonally dominant matrix $B([1:k,p],[1:k,p])$ produces also a zero jth pivot and the jth row is also entirely 0. Hence $(g_B)_{pp}^{(k+1)} = \det B([1:k,p],[1:k,p]) = 0$. □

Lemma 3.8 establishes a perturbation result for the nonprincipal minors defined in (2.8) under the standard perturbations defined in (3.2). This lemma is a consequence of the considerable effort we have done so far on studying structure perturbations of minors of row diagonally dominant matrices. Lemma 3.8 presents a different bound that the one in [11, Lemma 7(b)]. This will allow us to prove Theorem 3.2.

Lemma 3.8. *Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ be such that $v \geq 0$ and let $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}) \in \mathbb{R}^{n\times n}$ be a matrix that satisfies*

$$|\tilde{v} - v| \leq \epsilon v \quad \text{and} \quad |\tilde{A}_D - A_D| \leq \epsilon |A_D|, \quad \text{for some } 0 \leq \epsilon < \frac{1}{5}. \tag{3.32}$$

Let $1 \leq k \leq n-2$, $k+1 \leq p,q \leq n$, and $p \neq q$. Then, we have

(a) *If $g_{kk}^{(k)} \neq 0$, let $A^{(k+1)} = [a_{ij}^{(k+1)}] = \mathcal{D}(A_D^{(k+1)}, v^{(k+1)})$, with $v^{(k+1)} = [v_i^{(k+1)}]$, be the row diagonally dominant matrix with nonnegative diagonal entries obtained after k stages of Gaussian elimination have been performed on A. Then*

$$\left| \tilde{g}_{pq}^{(k+1)} - g_{pq}^{(k+1)} \right| \leq \frac{4}{3} \left((1+\epsilon_0)^{k+1} - 1 \right) \left(v_p^{(k+1)} + \left| a_{pq}^{(k+1)} \right| \right) g_{kk}^{(k)},$$

where $\epsilon_0 = \dfrac{6\epsilon}{1-\epsilon}$.

(b) *If $g_{kk}^{(k)} = 0$, then $\tilde{g}_{pq}^{(k+1)} = g_{pq}^{(k+1)} = 0$.*

Proof. Suppose $g_{kk}^{(k)} \neq 0$. Define B and $\tilde{B}$ as in Lemma 3.6. By (3.23)-(3.24)-(3.25), we can apply Lemma 3.5 to the minors of B and $\tilde{B}$ defined in (3.17) to obtain

$$\left| (\tilde{g}_B)_{pq}^{(k+1)} - (g_B)_{pq}^{(k+1)} \right| \leq \frac{4}{3} \left((1+3\delta)^{k+1} - 1 \right) (g_B)_{pp}^{(k+1)}, \tag{3.33}$$

with $\delta = 2\epsilon/(1-\epsilon)$. By the construction of B and $\tilde{B}$, we have

$$(\tilde{g}_B)_{pq}^{(k+1)} = \tilde{g}_{pq}^{(k+1)} \text{ and } (g_B)_{pq}^{(k+1)} = g_{pq}^{(k+1)},$$

and hence, from (3.33),

$$\left|\tilde{g}_{pq}^{(k+1)} - g_{pq}^{(k+1)}\right| \leq \frac{4}{3}\left((1+3\delta)^{k+1} - 1\right)(g_B)_{pp}^{(k+1)}. \tag{3.34}$$

Next, apply Lemma 3.7(a) to get

$$\left|\tilde{g}_{pq}^{(k+1)} - g_{pq}^{(k+1)}\right| \leq \frac{4}{3}\left((1+3\delta)^{k+1} - 1\right)\left(v_p^{(k+1)} + \left|a_{pq}^{(k+1)}\right|\right) g_{kk}^{(k)},$$

which proves part (a).

Next we prove part (b). We have seen in the proof of Lemma 3.7 that $g_{kk}^{(k)} = 0$ implies that one of the pivots $a_{jj}^{(j)}$ (for $1 \leq j \leq k$) in the Gaussian elimination for A must be 0. Since $A^{(j)}$ is still row diagonally dominant, the jth row of $A^{(j)}$ must be entirely 0. Then applying $j-1$ stages of the Gaussian elimination to $A([1:k,p],[1:k,q])$ produces a jth row which is entirely 0 and hence $g_{pq}^{(k+1)} = 0$. Furthermore, by equation (3.14) in Lemma 3.4, we have that $g_{kk}^{(k)} = 0$ implies $\tilde{g}_{kk}^{(k)} = 0$ and the same argument we have used above for A can be used on $\tilde{A}$ to prove $\tilde{g}_{pq}^{(k+1)} = 0$. ◻

The results presented so far in Section 3.1 are valid for general row diagonally dominant matrices with nonnegative diagonal entries. From now on, we assume that the unperturbed matrix A is arranged for column diagonal dominance pivoting. This allows us to bound the sum of the absolute values of the entries below the diagonal of each column of the L factor in terms of the diagonally dominant parts of the corresponding Schur complement and the corresponding pivot, as seen in Lemma 3.9.

Lemma 3.9. *Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ be such that $v \geq 0$ and assume that A is arranged for the column diagonal dominance pivoting strategy. Let $A^{(k)} = \mathcal{D}(A_D^{(k)}, v^{(k)})$, with $A^{(k)} = [a_{ij}^{(k)}]$ and $v^{(k)} = [v_i^{(k)}]$, be the row diagonally dominant matrix with nonnegative diagonal entries obtained after $k-1$ stages of Gaussian elimination have been applied on A. Then, for $k \leq \operatorname{rank}(A)$, we have*

$$\sum_{i=k+1}^{n}\left(|a_{ik}^{(k)}| + v_i^{(k)}\right) \leq (n-k)\, a_{kk}^{(k)}.$$

Proof. According to (2.6), define $\delta_i^{(k)} := a_{ii}^{(k)} - \sum\limits_{j=k, j\neq i}^{n} |a_{ji}^{(k)}|$. Then, we have

$$a_{kk}^{(k)} = \max_{k\leq i\leq n}\left\{a_{ii}^{(k)} : \delta_i^{(k)} \geq 0\right\}.$$

If $\delta_i^{(k)} \geq 0$ for all $i = k+1, \ldots, n$, then

$$\sum_{i=k+1}^{n}\left(|a_{ik}^{(k)}| + v_i^{(k)}\right) \leq \sum_{i=k+1}^{n} a_{ii}^{(k)} \leq \sum_{i=k+1}^{n} a_{kk}^{(k)} \leq (n-k)a_{kk}^{(k)},$$

which proves the result in this case. Otherwise, if there is at least one $\delta_i^{(k)} < 0$ for some $i = k+1, \ldots, n$, then from the definition of $v_i^{(k)}$ we obtain for $k+1 \leq i \leq n$

$$|a_{ik}^{(k)}| + v_i^{(k)} = a_{ii}^{(k)} - \sum_{j=k+1, j\neq i}^{n} |a_{ij}^{(k)}|$$

and sum over i to obtain,

$$\begin{aligned}
\sum_{i=k+1}^{n}\left(|a_{ik}^{(k)}|+v_i^{(k)}\right) &= \sum_{i=k+1}^{n} a_{ii}^{(k)} - \sum_{i=k+1}^{n}\sum_{j=k+1,j\neq i}^{n} |a_{ij}^{(k)}| = \sum_{i=k+1}^{n} a_{ii}^{(k)} - \sum_{i=k+1}^{n}\sum_{j=k+1,j\neq i}^{n} |a_{ji}^{(k)}| \\
&= \sum_{i=k+1}^{n}\left(a_{ii}^{(k)} - \sum_{j=k+1,j\neq i}^{n} |a_{ji}^{(k)}|\right) = \sum_{i=k+1}^{n}\left(a_{ii}^{(k)} - \sum_{j=k,j\neq i}^{n} |a_{ji}^{(k)}| + |a_{ki}^{(k)}|\right) \\
&= \sum_{i=k+1}^{n}\left(\delta_i^{(k)} + |a_{ki}^{(k)}|\right) \leq a_{kk}^{(k)} + \sum_{i=k+1}^{n} \delta_i^{(k)} \\
&\leq a_{kk}^{(k)} + \sum_{i=k+1,\delta_i^{(k)}\geq 0}^{n} \delta_i^{(k)} \leq a_{kk}^{(k)} + \sum_{i=k+1,\delta_i^{(k)}\geq 0}^{n} a_{ii}^{(k)} \\
&\leq a_{kk}^{(k)} + \sum_{i=k+1,\delta_i^{(k)}\geq 0}^{n} a_{kk}^{(k)} \leq a_{kk}^{(k)} + (n-k-1)\, a_{kk}^{(k)} \\
&= (n-k)\, a_{kk}^{(k)},
\end{aligned}$$

since for $\delta_i^{(k)} \geq 0, \delta_i^{(k)} \leq a_{ii}^{(k)} \leq a_{kk}^{(k)}$. □

Finally, we are now ready to present the proof of Theorem 3.2.

Proof of Theorem 3.2. As a consequence of Theorem 3.1(a), we have $\mathrm{rank}(A) = \mathrm{rank}(\tilde{A})$. Therefore, from Definition 2.4, it is observed that we only need to pay attention to the variation of the strictly lower triangular entries of L in its first $r := \mathrm{rank}(A)$ columns. Using (2.2), Lemma 3.8(a) with $p = i$, $q = j$ and $k = j-1$, and [11, Lemma 4(b)], we have for $i > j$ and $1 \leq j \leq r$

$$\tilde{l}_{ij} = \frac{\tilde{g}_{ij}^{(j)}}{\tilde{g}_{jj}^{(j)}} = \frac{g_{ij}^{(j)} + \frac{4}{3}\chi(v_i^{(j)} + |a_{ij}^{(j)}|)g_{j-1,j-1}^{(j-1)}}{g_{jj}^{(j)}(1+\xi_1)\cdots(1+\xi_j)}, \tag{3.35}$$

where $|\xi_1| \leq \epsilon, \ldots, |\xi_j| \leq \epsilon$, and $|\chi| \leq ((1+\epsilon_0)^j - 1)$. Define

$$\zeta := \frac{1}{(1+\xi_1)\cdots(1+\xi_j)} - 1$$

and note $|\zeta| \leq \dfrac{1}{(1-\epsilon)^j} - 1$. Hence, from (3.35) and (2.5),

$$\tilde{l}_{ij} = \left(l_{ij} + \frac{\frac{4}{3}\chi(v_i^{(j)} + |a_{ij}^{(j)}|)}{a_{jj}^{(j)}}\right)(1+\zeta) \quad\text{and}\quad \tilde{l}_{ij} - l_{ij} = \zeta l_{ij} + \frac{\frac{4}{3}\chi(1+\zeta)(v_i^{(j)} + |a_{ij}^{(j)}|)}{a_{jj}^{(j)}}.$$

Taking the absolute value gives

$$|\tilde{l}_{ij} - l_{ij}| \leq |\zeta||l_{ij}| + \frac{4}{3}|\chi||1+\zeta|\frac{(v_i^{(j)} + |a_{ij}^{(j)}|)}{a_{jj}^{(j)}},$$

and then summing over i yields

$$\sum_{i=j+1}^{n} |\tilde{l}_{ij} - l_{ij}| \leq |\zeta| \sum_{i=j+1}^{n} |l_{ij}| + \frac{4}{3}|\chi||1+\zeta| \frac{\sum_{i=j+1}^{n}(v_i^{(j)} + |a_{ij}^{(j)}|)}{a_{jj}^{(j)}}. \tag{3.36}$$

By assumption A is arranged for column diagonal dominance pivoting, which means that the matrix L is column diagonally dominant, that is, $\sum_{i=j+1}^{n} |l_{ij}| \leq 1$ for all j. Use this fact and Lemma 3.9 in (3.36) to get, for $1 \leq j \leq r$,

$$\begin{aligned}\sum_{i=j+1}^{n} |\tilde{l}_{ij} - l_{ij}| &\leq |\zeta| + \frac{4}{3}|\chi||1+\zeta| \, \frac{(n-1)a_{jj}^{(j)}}{a_{jj}^{(j)}} = |\zeta| + \frac{4}{3}(n-1)|\chi||1+\zeta| \\ &\leq \left(\frac{1}{(1-\epsilon)^n} - 1\right) + \frac{4}{3}(n-1)\left((1+\epsilon_0)^n - 1\right) \, \frac{1}{(1-\epsilon)^n} \\ &\leq \frac{n(4n-1)\epsilon_0}{3(1-2n\epsilon_0)} = \frac{2\,n\,(4n-1)\,\epsilon}{1-(12\,n+1)\,\epsilon}\,,\end{aligned}$$

where we have used $\epsilon < \epsilon_0 = 6\epsilon/(1-\epsilon)$, standard results from [20, Ch. 3], and $(12\,n+1)\,\epsilon < 1$. Theorem 3.2 follows from observing that

$$\|\tilde{L} - L\|_1 = \max_{1 \leq j \leq (n-1)} \sum_{i=j+1}^{n} |\tilde{l}_{ij} - l_{ij}|. \quad \square$$

4. Conclusions and future work. We have proved that small relative perturbations in the diagonally dominant parts and off-diagonal entries of row diagonally dominant matrices with nonnegative diagonal entries produce small relative normwise perturbations in the L factor obtained by applying the column diagonal dominance pivoting strategy to this type of matrices. This result can be combined with the perturbation results for the D and U factors presented in [11] to prove that the column diagonal dominance pivoting strategy for row diagonally dominant matrices leads, simultaneously, to LDU factorizations that are guaranteed to be rank revealing decompositions, i.e., the factors L and U are guaranteed to have small condition numbers, and that always undergo small relative perturbations under small relative perturbations in the diagonally dominant parts and off-diagonal entries. The perturbation results presented in this paper are fundamental to prove in [8] that essentially all interesting magnitudes corresponding to row diagonally dominant matrices undergo small relative perturbations under small relative perturbations in the diagonally dominant parts and off-diagonal entries and, therefore, that these magnitudes can be computed with high accuracy by algorithms based on rank revealing decompositions [5, 9, 12, 14, 33].

REFERENCES

[1] A. S. Alfa, J. Xue, and Q. Ye, *Entrywise perturbation theory for diagonally dominant M-matrices with applications*, Numer. Math., 90 (2002), pp. 401–414.

[2] A. Barrlund, *Perturbation bounds for the LDL^H and LU decompositions*, BIT, 31 (1991), pp. 358–363.

[3] S. Bora, *Structured eigenvalue condition number and backward error of a class of polynomial eigenvalue problems*, SIAM J. Matrix Anal. Appl., 31 (2009), pp. 900–917.

[4] S. Bora and V. Mehrmann, *Linear perturbation theory for structured matrix pencils arising in control theory*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 148–169.

[5] N. Castro-González, J. Ceballos, F. M. Dopico, and J. M. Molera, *Accurate solution of structured least squares problems via rank-revealing decompositions*, SIAM J. Matrix Anal. Appl., 34 (2013), pp. 1112–1128.

[6] X.-W. Chang and R.-C. Li, *Multiplicative perturbation analysis for QR factorizations*, Numer. Algebra Control Optim., 1 (2011), pp. 301–316.

[7] X.-W. Chang and C. C. Paige, *On the sensitivity of the LU factorization*, BIT, 38 (1998).

[8] M. Dailey, F. M. Dopico, and Q. Ye, *Relative perturbation theory for diagonally dominant matrices*. In preparation, 2013.
[9] J. Demmel, M. Gu, S. Eisenstat, I. Slapničar, K. Veselić, and Z. Drmač, *Computing the singular value decomposition with high relative accuracy*, Linear Algebra Appl., 299 (1999), pp. 21–80.
[10] F. M. Dopico and P. Koev, *Accurate symmetric rank revealing and eigendecompositions of symmetric structured matrices*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 1126–1156.
[11] F. M. Dopico and P. Koev, *Perturbation theory for the LDU factorization and accurate computations for diagonally dominant matrices*, Numer. Math., 119 (2011), pp. 337–371.
[12] F. M. Dopico, P. Koev, and J. M. Molera, *Implicit standard Jacobi gives high relative accuracy*, Numer. Math., 113 (2009), pp. 519–553.
[13] F. M. Dopico and J. M. Molera, *Perturbation theory for factorizations of LU type through series expansions*, SIAM J. Matrix Anal. Appl., 27 (2005), pp. 561–581.
[14] F. M. Dopico and J. M. Molera, *Accurate solution of structured linear systems via rank-revealing decompositions*, IMA J. Numer. Anal., 32 (2012), pp. 1096–1116.
[15] F. R. Gantmacher, *The Theory of Matrices*, Chelsea, New York, 1959.
[16] G. Golub and C. Van Loan, *Matrix Computations*, Johns Hopkins University Press, Baltimore, MD, 3rd ed., 1996.
[17] D. J. Higham and N. J. Higham, *Backward error and condition of structured linear systems*, SIAM J. Matrix Anal. Appl., 13 (1992), pp. 162–175.
[18] ———, *Structured backward error and condition of generalized eigenvalue problems*, SIAM J. Matrix Anal. Appl., 20 (1999), pp. 493–512.
[19] N. J. Higham, *A survey of componentwise perturbation theory in numerical linear algebra*, in Mathematics of Computation 1943–1993: A Half Century of Computational Mathematics, 1994, pp. 49–77.
[20] N. J. Higham, *Accuracy and Stability of Numerical Algorithms*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second ed., 2002.
[21] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1994. Corrected reprint of the 1991 original.
[22] I. C. F. Ipsen, *Relative perturbation results for matrix eigenvalues and singular values*, in Acta Numerica, 1998, vol. 7 of Acta Numer., Cambridge Univ. Press, Cambridge, 1998, pp. 151–201.
[23] M. Karow, D. Kressner, and F. Tisseur, *Structured eigenvalue condition numbers*, SIAM J. Matrix Anal. Appl., 28 (2006), pp. 1052–1068.
[24] R.-C. Li, *A bound on the solution to a structured Sylvester equation with an application to relative perturbation theory*, SIAM J. Matrix Anal. Appl., 21 (1999), pp. 440–445.
[25] R.-C. Li, Y. Nakatsukasa, N. Truhar, and S. Xu, *Perturbation of partitioned Hermitian definite generalized eigenvalue problems*, SIAM J. Matrix Anal. Appl., 32 (2011), pp. 642–663.
[26] J. M. Peña, *LDU decompositions with L and U well conditioned*, Electron. Trans. Numer. Anal., 18 (2004), pp. 198–208.
[27] S. M. Rump, *Eigenvalues, pseudospectrum and structured perturbations*, Lin. Alg. Appl., 413 (2006), pp. 567–593.
[28] I. Slapničar, *Componentwise analysis of direct factorization of real symmetric and Hermitian matrices*, Linear Algebra Appl., 272 (1998), pp. 227–275.
[29] G. W. Stewart and J.-G. Sun, *Matrix Perturbation Theory*, Academic Press, New York, 1990.
[30] J. Sun, *Componentwise perturbation bounds for some matrix decompositions*, BIT, 32 (1992), pp. 702–714.
[31] F. Tisseur and N. J. Higham, *Structured pseudospectra for polynomial eigenvalue problems, with applications*, SIAM J. Matrix Anal. Appl., 23 (2001), pp. 187–208.
[32] K. Veselić and I. Slapničar, *Floating point perturbations of Hermitian matrices*, Lin. Alg. Appl., 195 (1993), pp. 81–116.
[33] Q. Ye, *Computing singular values of diagonally dominant matrices to high relative accuracy*, Math. Comp., 77 (2008), pp. 2195–2230.
[34] Q. Ye, *Relative perturbation bounds for eigenvalues of symmetric positive definite diagonally dominant matrices*, SIAM J. Matrix Anal. Appl., 31 (2009), pp. 11–17.