Inverse Boundary Value Problems for Polyharmonic Operators with Non-Smooth Coefficients

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AGAPI days X

Joint work

The results discussed today are joint work with a former student, Landon Gauthier [8].

Outline

Polyharmonic Operator

We fix a domain $\Omega \subset \mathbf{R}^d$, $d \geq 3$ with nice boundary and let $m \geq 2$. We consider the problem of identifying coefficients in a partial differential operator from boundary information from solutions of that operator.

The operators we consider are of the form

$$\mathcal{L}u = (-\Delta)^m u + Q \cdot Du + qu \qquad \text{in } \Omega \tag{1}$$

where $m \geq 2$ and $D = -i\nabla$. Our main result shows that the coefficients Q and q are uniquely determined by boundary information for solutions of $\mathcal{L}u = 0$.

The novel aspect of this work is to consider this problem for less regular coefficients.

Bilinear form

To define the operator $(-\Delta)^m$, we will use a weak formulation. If $u \in W^{m,2}(\Omega)$, we can define $(-\Delta)^m u \in W^{-m,2}(\Omega) = (W_0^{m,2}(\Omega))^*$ by

$$\langle (-\Delta)^m u, v\rangle = B_0(u,v), \quad v \in W_0^{m,2}(\Omega).$$

where $B_0 : W^{m,2}(\Omega) \times W^{m,2}(\Omega) \to \mathbf{C}$ is a bilinear form so that $B_0(u,v) = \int_\Omega [(-\Delta)^m u] v \, dx$, $u, v \in C_0^\infty(\Omega)$. For this talk we put

$$B_0(u,v) = \begin{cases} \int_\Omega (-\Delta)^{m/2} u \, (-\Delta)^{m/2} v \, dx, & m \text{ even} \\ \int_\Omega (-\Delta)^{\frac{m-1}{2}} \nabla u \cdot (-\Delta)^{\frac{m-1}{2}} \nabla v \, dx, & m \text{ odd} \end{cases}$$

though other choices are possible (and useful).

Coefficients

We will consider coefficients from Sobolev spaces $\tilde{W}^{s,p}(\Omega)$ (s and p will be fixed later). The space $\tilde{W}^{s,p}(\Omega)$ is the collection distributions in $W^{s,p}(\mathbf{R}^m)$ which are supported in $\bar{\Omega}$. Our main interest is when $s < 0$. This space arises as a dual space. In particular, for Lipschitz domains, we have

$$W^{s,p}(\Omega)^* = \tilde{W}^{-s,p'}(\Omega), \quad s \geq 0, 1 < p < \infty.$$

Suppose $\begin{cases} t = d/m, & d/m > 2, \\ t > 2, & d/m = 2 \\ t = 2, & d/m < 2. \end{cases}$

If $q \in \tilde{W}^{|\alpha|-m,t}(\Omega)$, then using Sobolev embedding, we can show

$$|\langle qD^\alpha u, v\rangle| \leq C\|q\|_{\tilde{W}^{|\alpha|-m,t}(\Omega)}\|u\|_{W^{m,2}(\Omega)}\|v\|_{W^{m,2}(\Omega)}.$$

Weak solutions

If $Q \in \tilde{W}^{1-m,t}(\Omega)$ and $q \in \tilde{W}^{-m,t}(\Omega)$, and $f \in W^{-m,2}(\Omega) \supset \tilde{W}^{-m,2}(\Omega)$, we say that u is a weak solution of $\mathcal{L}u = f$ if

$$\begin{aligned} B(u,v) &= B_0(u,v) + \langle Q \cdot Du, v\rangle + \langle qu, v\rangle \\ &= \langle f, v\rangle, \qquad v \in W_0^{m,2}(\Omega). \end{aligned}$$

While we only need to allow $v \in W_0^{m,2}(\Omega)$ to define solutions of $\mathcal{L}u = 0$, our form B is defined on $W^{m,2}(\Omega) \times W^{m,2}(\Omega)$.
We will say u a solution of the transposed equation $\mathcal{L}^t u = f$ if

$$B(v,u) = \langle f, v\rangle, \quad v \in W_0^{m,2}(\Omega).$$

Quadratic, forms, DN maps

If u is a solution of $\mathcal{L}u = 0$, then we may define Λu as a linear functional on the quotient space $W^{m,2}(\Omega)/W_0^{m,2}(\Omega)$ by

$$\langle \Lambda u, v \rangle = B(u, v).$$

Since u is a solution, the $v \to \langle \Lambda u, v \rangle$ map is well-defined on the equivalence classes $v + W_0^{m,2}(\Omega)$.
If we have two operators $\mathcal{L}_j$, $j = 1, 2$, we say that the corresponding forms B_1 and B_2 are equal if for each u_1, a solution of $\mathcal{L}_1 u_1 = 0$, there exists u_2, a solution of $\mathcal{L}_2 u_2 = 0$ with $u_1 - u_2 \in W_0^{m,2}(\Omega)$ and

$$B_1(u_1, v) = B_2(u_2, v), \quad v \in W^{m,2}(\Omega).$$

Also, the corresponding statement holds with 1 and 2 switched.

A key identity

If u_1 solves $\mathcal{L}_1 u_1 = 0$ and u_1, u_2 are in $W^{m,2}(\Omega)$, then our assumption that the forms are equal implies there exists a solution of $\mathcal{L}_2 \tilde{u}_2 = 0$ with $u_1 - \tilde{u}_2 \in W_0^{m,2}(\Omega)$ and $B_1(u_1, u_2) = B_2(\tilde{u}_2, u_2)$. If we also assume that $\mathcal{L}_2^t u_2 = 0$, $B_2(u_1 - \tilde{u}_2, u_2) = 0$. Combining these statements gives $B_1(u_1, u_2) = B_2(u_1, u_2)$. Dropping the common terms we are left with

$$\langle (q^1 - q^2) u_1, u_2 \rangle + \langle (Q^1 - Q^2) \cdot D u_1, u_2 \rangle = 0, \quad \text{if } \mathcal{L}_1 u_1 = 0, \mathcal{L}_2^t u_2 = 0.$$

This is often called Alessandrini's identity [1].
The proof of uniqueness rests on finding enough solutions for this identity to imply that $q^1 = q^2$ and $Q^1 = Q^2$.

Cauchy data, Dirichlet to Neumann maps

Our proof below will work directly with the quadratic forms. However, if the domain and the coefficients are sufficiently regular, we can reformulate the statement that two quadratic forms are equal as a statement about Cauchy data for solutions. If we also have unique solvability of the Dirichlet problem for the operators $\mathcal{L}_j$, then we may restate the equality of forms using a Dirichlet to Neumann map.

Main Theorem

Our main result is a statement that the if two operators have the same bilinear form (as defined above), then the coefficients must be equal.

Theorem

Suppose that we have two operators $\mathcal{L}_j = (-\Delta)^m + Q^j \cdot D + q^j$, $j = 1, 2$ as in (1) and we have $s < m/2 + 1$ and $p \geq 2$ which satisfy $1/p + (s - m)/d < 0$, $p \geq 2$. Suppose the coefficients of the operators $\mathcal{L}_j$ satisfy $Q^j \in \tilde{W}^{-s+1,p}(\Omega; \mathbf{R}^d)$ and $q^j \in \tilde{W}^{-s,p}(\Omega)$. If the bilinear forms B_j for the operators are equal, then $Q^1 = Q^2$ and $q^1 = q^2$.

Harmonic exponentials

The first step in constructing solutions to identify the coefficients is an observation of Calderón [9]. In 1980, he considered a linearized problem when the principal part is the Laplacian ($m = 1$) and gave an argument which used harmonic exponentials

$$e^{ix\cdot\zeta}, \quad \text{with } \zeta\cdot\zeta = 0.$$

In 1987, Sylvester and Uhlmann [14] carried out the construction of solutions which are close to harmonic exponentials and showed how to use them to prove uniqueness for an inverse boundary value problem for a second order equation.

CGO solutions

The solutions constructed by Sylvester and Uhlmann are now called complex geometrical optics (CGO) solutions. For our purposes, we will look at CGO solutions of the form

$$u(x) = e^{ix\cdot\zeta/h}(a(x) + \psi(x))$$

where the remainder $\psi(x) = \psi(x; a, \zeta, h)$ will be small in an appropriate sense as $h \to 0^+$.

We assume that $h > 0$ is a small parameter and we will use $\zeta \in \mathcal{V} = \{\zeta \in \mathbf{C}^d, |\,\mathrm{Re}\,\zeta| = |\,\mathrm{Im}\,\zeta| = 1, \mathrm{Re}\,\zeta \cdot \mathrm{Im}\,\zeta = 0\}$ so that $e^{ix\cdot\zeta/h}$ will be a harmonic function when $\zeta \in \mathcal{V}$.

X^λ spaces

We let $p_\zeta(hD) = p(hD) = e^{-ix\cdot\zeta/h}(-h^2\Delta)e^{ix\cdot\zeta/h}$ and denote the symbol of $p(hD)$ by $p(\xi) = |\xi|^2 + 2i\zeta\cdot\xi$.
For $\lambda \in \mathbf{R}$ and $\zeta \in \mathcal{V}$, we define

$$X^\lambda_{h\zeta} = X^\lambda = \{u : \|u\|^2_{X^\lambda} = \int_{\mathbf{R}^d} (h + |p(h\xi)|)^{2\lambda} |\hat{u}(\xi)|^2 \, d\xi < \infty\}. \tag{2}$$

We will often drop the subscript if h and ζ are clear from the context.
These spaces were introduced to the study of inverse boundary value problems by Haberman and Tataru [11] in 2013. They borrowed ideas from Bourgain who had used spaces adapted to the operator $\partial_t + \partial_x^3$ in the study of the Korteweg-deVries equation.

A right inverse

We let $\phi \in C_c^\infty(\mathbf{R}^d)$ be function which is 1 in a neighborhood of $\bar{\Omega}$ and define an operator J_ϕ by

$$J_\phi f = \phi \cdot \left(\frac{1}{p(h\xi)}(\phi f)\hat{}\right)\check{}$$

Unraveling the notation, it is easy to see that in a neighborhood of $\bar{\Omega}$, we have J_ϕ is a right inverse of $p(hD)$:

$$p(hD)J_\phi f = f.$$

Building on ideas of Haberman and Tataru, we can show

$$J_\phi : X^\lambda \to X^{\lambda+1}, \quad \lambda \in \mathbf{R}.$$

The details can be found in Brown and Gauthier [8].

Multiplication in the X^{λ}-spaces

To study the operator $\mathcal{L}$, we will need several results about the multiplication operator

$$\psi \to q\psi$$

on the spaces X^{λ}.

Our first observation is that multiplying by a smooth function is bounded on the space X^{λ}.

Lemma

If $\phi \in C^{\infty}(\mathbf{R}^d)$ then for each λ, there is $M = M(\lambda)$ so that

$$\|\phi u\|_{X^{\lambda}} \leq C(\sup_{x\in\mathbf{R}^d, |\alpha|\leq M(\lambda)} |D^{|\alpha|}\phi|)\|u\|_{X^{\lambda}}.$$

Multiplying by distributions

Next, we observe a relation between the X^λ-spaces and L^2 Sobolev spaces.

If $0 \leq 2s \leq t$, then

$$\|f\|_{W^{s,2}(\mathbf{R}^d)} \leq \sup \frac{\langle \xi \rangle^s}{(h + |p(h\xi)|)^t} \|f\|_{X^t} = Ch^{-t-s}\|f\|_{X^t}$$

Passing to the duals, we also have

$$\|f\|_{X^{-t}} \leq Ch^{-t-s}\|f\|_{W^{-s,2}(\mathbf{R}^d)}$$

Using this, we can show that if $f \in L^\infty(\mathbf{R}^d)$ and $|\alpha| + |\beta| \leq 2\lambda$, then

$$|\langle (D^\alpha f) D^\beta u, v\rangle| \leq Ch^{-2\lambda-|\alpha|-|\beta|}\|f\|_\infty \|u\|_{X^\lambda_{h\zeta_1}} \|v\|_{X^\lambda_{h\zeta_2}} \qquad (3)$$

If f is uniformly continuous, we have $C = o(1)$ as $h \to 0^+$

Defining $\mathcal{L}$ on X^λ

We assume that q and Q are of the form,

$$q = \sum_{|\alpha| \leq m} D^\alpha f_\alpha, \quad Q = \sum_{\alpha \leq m-1} D^\alpha g_\alpha \tag{4}$$

with the functions f_α and g_α uniformly continuous on $\mathbf{R}^d$. With these representations, we can show that the operator A given by

$$A\psi = q\psi + Q \cdot (D + \frac{\zeta}{h})\psi$$

satisfies

$$\|A\psi\|_{X^{-m/2}} \leq o(h^{-2m})\|\psi\|_{X^{m/2}}$$

Representations of q and Q

Using ideas of Mitrea[2] and Monniaux [13] (which is based on work of Bogovskii [7]) we can establish the following representation theorem.

If $f \in \tilde{W}^{-s,p}(\Omega)$ with $-s = -k + \sigma$, $0 \leq \sigma < 1$ and $k \geq 1$ an integer, then we may write

$$f = \sum_{|\alpha| \leq k} D^\alpha f_\alpha, \qquad f_\alpha \in \tilde{W}^{\sigma,p}(\Omega)$$

Using Sobolev embedding and this representation, if $f \in \tilde{W}^{-s,p}(\Omega)$ with $1/p + (s-m)/d < 0$, we may write

$$f = \sum_{|\alpha| \leq m} D^\alpha f_\alpha$$

with $f_\alpha \in C(\mathbf{R}^d)$ and $\operatorname{supp} f \subset \bar{\Omega}$.

Existence of solutions

The following theorem will be essential for our construction of CGO solutions. Let

$$\begin{aligned} \mathcal{L}_\zeta \psi &= h^{2m} e^{-ix\cdot\zeta/h} \mathcal{L} e^{ix\cdot\zeta/h} \\ &= p(hD)^m \psi + h^{2m}(q\psi + Q\cdot(D+\zeta/h)\psi) \\ &= p(hD)^m\psi + h^{2m}A\psi \end{aligned}$$

Theorem (Existence of solutions)

Suppose q and Q are as in (4), then for $\zeta \in \mathcal{V}$, h small, and $f \in X^{-m/2}$ we may find $\psi \in X^{m/2}$ so that ψ is a distribution solution of

$$\mathcal{L}_\zeta \psi = f \quad \text{in a neighborhood of } \bar{\Omega}$$

$$\|\psi\|_{X^{m/2}} \leq C\|f\|_{X^{-m/2}}$$

Solve $\mathcal{L}_\zeta \psi = f$

Proof.

We observe that a solution of the integral equation

$$\psi + h^{2m} J_\phi^m A \psi = J_\phi^m \phi f$$

will also solve $\mathcal{L}_\zeta \psi = f$ on Ω.
Since $J_\phi : X^\lambda \to X^{\lambda+1}$ and $A : X^{m/2} \to X^{-m/2}$, for h small, we have that $h^{2m} J_\phi^m A$ is a contraction on $X^{m/2}$ and we may use a Neumann series to invert the operator $I + h^{2m} J_\phi^m A$ on $X^{m/2}$. □

CGO solutions

We will look for solutions of $\mathcal{L}u = 0$ of the form

$$u(x) = e^{ix\cdot\zeta/h}(a(x) + \psi(x))$$

we need ψ to satisfy the equation

$$\mathcal{L}_\zeta \psi = -p(hD)a - h^{2m}(qa + Q \cdot (D + \zeta/h)a).$$

Amplitudes

Our amplitudes a will be smooth functions which satisfy the transport equation

$$(\zeta \cdot D)^2 a = 0.$$

This will guarantee that we have $D^\alpha p(hD)^m a = O(h^{2m-1})$ on compact subsets of $\mathbf{R}^d$. Thus for a cutoff function ϕ, we have

$$\|\phi(p(hD))^m a\|_{X^{-m/2}} = O(h^{\frac{3m}{2}-1}).$$

More specifically, the amplitudes we use are

$$a(x) = (\alpha + \beta \cdot x)e^{-ix\cdot\xi}$$

where we have $\xi \cdot \zeta = 0$ and $\alpha + \beta \cdot x$ is a linear function.

Estimate for the right-hand side

If in addition, we have $p \geq 2$, then $\tilde W^{-s,p}(\Omega) \subset \tilde W^{-s,2}(\Omega)$ and for $0 \leq s \leq m$

$$\begin{aligned}\|aq + Q \cdot (D + \frac{\zeta}{h})a\|_{X^{-m/2}} &\leq C\|q\|_{\tilde W^{-s,2}(\Omega} \sup \frac{\langle \xi \rangle^s}{(h + |p(h\xi)|)^{m/2}} \\ &\quad + h^{-1}\|Q\|_{W^{1-s,2}(\Omega)} \sup \frac{\langle \xi \rangle^{s-1}}{(h + |p(h\xi)|)^{m/2}} \\ &\leq Ch^{-s-m/2}(\|q\|_{\tilde W^{-s,2}(\Omega)} + \|Q\|_{\tilde W^{1-s,2}(\Omega)}).\end{aligned}$$

Elements of the proof of uniqueness

We fix $\xi \in \mathbf{R}^d$, set $a(x) = e^{-ix\cdot\xi}$, assume $\zeta \in \mathcal{V}$ satisfies $\xi \cdot \zeta = 0$ so that $\zeta \cdot De^{-ix\cdot\xi} = 0$. We let ψ be the solution of

$$\mathcal{L}_\zeta \psi = -\phi \mathcal{L}_\zeta a = -\phi(p(hD)^m a + h^{2m}(q + Q \cdot (D + \zeta/h)a).$$

where $\phi \in \mathbf{C}_c^\infty(\mathbf{R}^d)$ is 1 in a neighborhood of $\bar{\Omega}$. Using our estimates for $p(hD)^m a$ and $(q + Q \cdot (\zeta/h + D))a$ we have

$$\|\psi\|_{X^{m/2}} \leq C(h^{\frac{3m}{2}-s} + h^{\frac{3m}{2}-1}).$$

First uniqueness theorem

We construct solutions of $\mathcal{L}_1 u_1 = 0$, $\mathcal{L}_2^t u_2 = 0$ with the expansions

$$u_1(x) = e^{ix\cdot\zeta/h}(e^{-ix\cdot\xi} + \psi_1), \quad u_2(x) = e^{-ix\cdot\zeta/h}(1 + \psi_2)$$

Using that our forms B_1 and B_2 are equal, we have

$$\begin{aligned} 0 &= B_1(u_1, u_2) - B_2(u_1, u_2) \\ &= \hat{q}^1(\xi) - \hat{q}^2(\xi) + \langle (q^1 - q^2), (\psi_1 + e^{-ix\cdot\xi}\psi_2 + \psi_1\psi_2)\rangle + \dots \end{aligned}$$

where we have omitted several terms in order to allow us to concentrate on the main idea in this proof.

A hint of the uniqueness proof

We have the estimates

$$|\langle (q^1 - q^2)a, \psi_j\rangle| \leq \|q^1 - q^2\|_{X^{-m/2}}\|\psi_j\|_{X^{m/2}}$$
$$= h^{-m/2-s}(h^{3m/2-1} + h^{3m/2-s}$$

$$\langle (q^1 - q^2)\psi_1, \psi_2\rangle \leq o(1)h^{-2m}(h^{3m-1} + h^{3m-2s})$$

If $s \leq m/2$ (and $m \geq 2$), we conclude $\hat{q}_1 - \hat{q}_2 = 0$.
A more involved argument, which we omit, allows us to show $Q^1 - Q^2 = 0$.

Some history

An inverse boundary value problem for biharmonic operators was first studied by Krupchyk, Lassas, and Uhlmann in a paper that appeared in 2014. Additional work considers a problem with partial data [12].

Stability in this inverse boundary value problem was considered by Choudhury and Krishnan [10].

In 2016 Assylbekov [4] and in 2019 Assylbekov and Iyer [3] begin the study of inverse boundary value problems for polyharmonic operators with less regular coefficients. Their result is very close to our first uniqueness theorem.

Our approach their result using X^λ-spaces is new, but the result is not.

In 2019 and 2012, Bhattacharyya and Ghosh [5, 6] study uniqueness for second order terms.

In 2023, Aroua and Bellassoued [2] give some results on the stability of recovery of second order terms.

How can we do better?

The construction of CGO solutions involves an arbitrary choice of ζ from $\mathcal{V}_\xi = \mathcal{V} \cap \{\zeta : \xi \cdot \zeta = 0\}$, The set $\mathcal{V}_\xi$ is positive dimension $(2d - 5?)$ and thus there are many possible choices. We will use an averaging argument to take advantage of this freedom to improve on our first uniqueness result.

This approach was first developed by Haberman and Tataru [11] in a study of the inverse conductivity problem.

A key lemma

Again we fix ξ and choose $\zeta \in \mathcal{V}$ with $\xi \cdot \zeta = 0$.
If we set $\zeta(\theta) = e^{i\theta}\zeta$, we have the following Lemma.

Lemma

For $0 \leq \sigma < 1$ and $0 < h < 1$, we have

$$\frac{1}{2\pi h} \int_h^{2h} \int_0^{2\pi} \frac{1}{(\tau + |p_{\zeta(\theta)}(\tau\eta)|)^{2\sigma}} \, d\theta \, d\tau \leq \frac{1}{(h\langle\eta\rangle)^{4\sigma}}$$

If we fix η, the function $(h, \theta) \to p_{\zeta(\theta)}(\tau\eta)$ has a simple zero. This observation helps to see how we prove this result.

We apply the previous Lemma to obtain

Lemma

If $0 \leq \sigma < 1$ and $2\sigma < s$, then we have

$$\frac{1}{2\pi h}\int_0^{2\pi}\int_h^{2h} \|f\|^2_{X^{-m/2}_{\tau\zeta(\theta)}}\, d\tau d\theta \leq C h^{-m-2s+2\sigma}\|f\|^2_{W^{-s,2}}$$

If $q^k \in W^{-s,2}(\mathbf{R}^d)$ and $Q^k \in \tilde{W}^{1-s,2}(\mathbf{R}^d)$, and $0 \leq \sigma < 1$ this lemma allows us to choose a sequence of ζ_j, h_j with $h_j \to 0$ and so that with

$$\|q^1\|_{X^{-m/2}_{h_j\zeta_j}} + \|q^2\|_{X^{-m/2}_{-h_j\zeta_j}} + h_j^{-1}(\|Q^1\|_{X^{(1-m)/2}_{h_j\zeta_j}} + \|Q^2\|_{X^{(1-m)/2}_{-h_j\zeta_j}}) \leq C h_j^{-m/2-s+\sigma}$$

Feeding this improvement for the power of h into our construction of CGO solutions leads to a uniqueness theorem for operators with the condition with $s < m/2 + 1$, rather than the condition $s \leq m/2$ from our first attempt.

Open Questions

- Can we reduce the regularity on q or Q? We conjecture that $q \in \tilde{W}^{-m,?}(\Omega)$ is the best order of smoothness. For $m = 2$, we get arbitrarily close to this result.
- Determine the correct index p for the L^p-space in these uniqueness results.
- Can we prove a uniqueness result for the polyharmonic operator with higher order perturbations? That is for operators of the form

$$\mathcal{L} = (-\Delta)^m + \sum_{|\alpha| \leq m-1} Q_\alpha D^\alpha$$

  are the coefficients Q_α determined by the form?

Thanks

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