Worksheet on induction | MA113:00
Calculus I | Fall 2005

First, let us explain the use of the $\sum$ for summation. The notation

$$\sum_{k=1}^{n} f(k)$$

means to evaluate the function $f(k)$ at $k = 1, 2, \ldots, n$ and add up the results. In other words:

$$\sum_{k=1}^{n} f(k) = f(1) + f(2) + \ldots + f(n).$$

For example:

$$\sum_{k=1}^{4} k^2 = 1 + 4 + 9 + 16,$$

$$\sum_{k=1}^{n} 2k - 1 = 1 + 3 + 5 + \ldots + 2n - 1,$$

and

$$\sum_{k=1}^{n} 1 = n.$$

The principle of mathematical induction is used to establish the truth of a sequence of statements or formula which depend on a natural number, $n = 1, 2, \ldots$. The principle is:

Principle of mathematical induction Suppose that P_n is a sequence of statements depending on a natural number $n = 1, 2, \ldots$. If we can show that:

- P_1 is true
- For $N = 1, 2, \ldots$: If P_N is true, then P_{N+1} is true.

Then, we can conclude that all the statements P_n are true.

The reason this works is that if we know P_1 is true, then the second step allows us to conclude P_2 is true. Now that we know P_2 is true, then the second step allows us to conclude P_3 is true. If we repeat this $n - 1$ times, we know that P_n is true. This principle is useful because it allows us to prove an infinite number of statements are true in just two easy steps!
Below are several examples to illustrate how to use this principle. In the examples below, you should note that, in the second step, the key point is to show how to go from the statement P_N to P_{N+1}.

Example Show that for $n = 1, 2, 3, \ldots$, the number $4^n - 1$ is a multiple of 3.

Solution Step 1. We need to show this is true when $n = 1$. This is easy since $4^1 - 1 = 4 - 1 = 3$ and 3 is divisible by 3.
Step 2. We suppose that $4^N - 1$ is a multiple of 3 and we want to use this assumption to show that $4^{N+1} - 1$ is a multiple of 3. Our assumption for N means that for some whole number M, $4^N - 1 = 3M$. Now $4^{N+1} - 1$. If we add and subtract 4, we have

$$4^{N+1} - 1 = 4^{N+1} - 4 + 4 - 1 = 4(4^N - 1) + 3.$$

Now we use our assumption that $4^N - 1$ is a multiple of 3 to replace $4^N - 1$ by $3M$ and obtain that

$$4^{N+1} - 1 = 4 \cdot 3M + 3 = 3(4M + 1).$$

Thus we have shown that $4^{N+1} - 1$ is a multiple of 3.

Example Show that for $n = 1, 2, \ldots$, we have

$$\sum_{j=1}^{n} 2j = n(n + 1).$$

Solution Step 1. If $n = 1$, then $n(n + 1) = 1 \cdot 2 = 2$. Also,

$$\sum_{j=1}^{1} 2j = 2.$$

Thus both sides are equal if $n = 1$.
Step 2. Now suppose that the formula is true for N and consider the sum

$$\sum_{j=1}^{N+1} 2j = \sum_{j=1}^{N} 2j + 2(N + 1).$$

We use our assumption that $\sum_{j=1}^{N} 2j = N(N + 1)$ to conclude that

$$\sum_{j=1}^{N+1} 2j = N(N + 1) + 2(N + 2).$$

Simplifying this last expression gives

$$N(N + 1) + 2(N + 1) = N^2 + N + 2N + 2 = N^2 + 3N + 2 = (N + 2)(N + 1).$$

Since $(N + 2)(N + 1) = (N + 1 + 1)(N + 1)$, we have shown that the formula

$$\sum_{j=1}^{N+1} 2j = (N + 1 + 1)(N + 1)$$

is true. This completes the proof by induction.

Below is a selection of problems related to mathematical induction. You should begin working on these problems in recitation. Write up your solutions carefully and elegantly and hand them in by the due date announced.

1. (a) For $n = 1, 2, 3, 4$, compute

$$\sum_{k=1}^{n} (2k - 1).$$

Make a guess for the value of this sum for $n = 1, 2, \ldots$.

(b) Use mathematical induction to prove that your guess is correct.

2. Use mathematical induction to prove that

$$\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6.$$

3. Let $f_1(x) = x - 2$ and then define f_n for $n = 1, 2, \ldots$ by $f_{n+1}(x) = f_1(f_n(x))$. (It is the principle of mathematical induction which tells us that these two statements suffice to define f_n for all n.) Use mathematical induction to prove that

$$f_n(x) = x - 2n.$$

4. Let P_n be the statement: $n^2 - n$ is an odd integer.

   (a) Show that if P_n is true, then P_{n+1} is true.

   (b) Is P_1 true?

   (c) Is P_n true for any n?

Below are some additional problems. You may not be able to solve all of these problems at this time. These problems will not be collected.

1. Let $f(x) = \sin(2x)$. Prove that for $n = 1, 2, \ldots$,

$$\frac{d^{2n}}{dx^{2n}} f(x) = (-4)^{2n} \sin(2x).$$

2. Let $f(x) = xe^x$. Compute f', f'', and f'''. Guess a formula for the nth derivative,

$$\frac{d^n}{dx^n} f(x).$$

Prove that your guess is right.

3. (a) Find a simple formula for

$$\sum_{k=1}^{n} (k+1)^2 - k^2 = 2^2 - 1 + (3^2 - 2^2) + \ldots + n^2 - (n-1)^2 + (n+1)^2 - n^2.$$

(b) Using your answer to part a), find a simple expression for

$$\sum_{k=1}^{n}(2k-1).$$

To do this you should simplify each summand on the left.

4. Use mathematical induction to prove that

$$\sum_{j=1}^{n} j^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

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