1 The fundamental theorems of calculus.

- The fundamental theorems of calculus.
- Evaluating definite integrals.
- The indefinite integral-a new name for anti-derivative.
- Differentiating integrals.

**Theorem 1** Suppose \( f \) is a continuous function on \([a, b]\).

**(FTC I)** If \( g(x) = \int_{a}^{x} f(t) \, dt \), then \( g'(x) = f(x) \).

**(FTC II)** If \( F \) is an anti-derivative of \( f \), then

\[
\int_{a}^{b} f(t) \, dt = F(b) - F(a).
\]

**Example.** Compute \( \frac{d}{dx} \int_{1}^{x} \frac{1}{t} \, dt \).

Compute \( \int_{0}^{3} x^3 \, dx \).

**Proof.** An idea of the proofs. FTC I:

Write

\[
\frac{g(x+h) - g(x)}{h} = \frac{1}{h} \int_{x}^{x+h} f(t) \, dt.
\]

We will show

\[
\lim_{h \to 0^+} \frac{1}{h} \int_{x}^{x+h} f(t) \, dt = f(x).
\]

The reader should write out a similar argument for the limit from the below.

If \( f \) is continuous, then \( f \) has maximum and minimum values \( M_h \) and \( m_h \) on the interval \([x, x+h]\). Using the order property of the integral,

\[
m_h \leq \frac{1}{h} \int_{x}^{x+h} f(t) \, dt \leq M_h.
\]

As \( h \) tends to 0, we have \( \lim_{h \to 0^+} M_h = \lim_{h \to 0^+} m_h = f(x) \) since \( f \) is continuous. It follows that

\[
\lim_{h \to 0^+} \frac{1}{h} \int_{x}^{x+h} f(t) \, dt = f(x).
\]

FTC II:
We know from FTCI that \( f \) has one anti-derivative, \( \int_a^x f(t) \, dt \). We let

\[
G(x) = \int_a^x f(t) \, dt - F(x)
\]

where \( F \) is some anti-derivative as in FTC II. The derivative of \( G \), \( G'(x) = f(x) - f(x) = 0 \) for all \( x \) in \( (a, b) \). This uses FTC I and the hypothesis that \( F \) is an anti-derivative of \( f \). Since the derivative of \( G \) is identically zero, we can conclude that \( G \) is a constant.

If we set \( x = a \) in the definition of \( G \), we find \( G(a) = -F(a) \) so that we can conclude the constant is \( -F(a) \). If we set \( x = b \) in the definition of \( G \), then we conclud

\[
-F(a) = \int_a^b f(t) \, dt - F(b).
\]

Adding \( F(b) \) to both sides give the conclusion of FTC II.

1.1 Indefinite integrals.

We use the symbol

\[
\int f(x) \, dx
\]

to denote the indefinite integral or anti-derivative of \( f \).

The indefinite integral is a function. The definite integral is a number. According FTC II, we can find the (numerical) value of a definite integral by evaluating the indefinite integral at the endpoints of the integral. Since this procedure happens so often, we have a special notation for this evaluation.

\[
F(x)|_{x=a}^{b} = F(b) - F(a).
\]

**Example.** Find

\[
xa|_{x=a}^{b} \quad \text{and} \quad xa|_{a=x}^{b}
\]

**Solution.**

\[
ba - a^2 \quad xy - x^2
\]

According to FTC I, anti-derivatives exist provided \( f \) is continuous.

The box on page 351 should be memorized. (In fact, you should already have memorized this information when we studied derivatives in Chapter 3 and when we studied anti-derivatives in Chapter 4.)

**Example.** Verify

\[
\int x \cos(x^2) \, dx = \frac{1}{2} \sin(x^2).
\]
Solution. According to the definition of anti-derivative, we need to see if
\[ \frac{d}{dx} \frac{1}{2} \sin(x^2) = x \cos(x^2). \]
This holds, by the chain rule.

\[ \square \]

1.2 Computing integrals.

The main use of FTC II is to simplify the evaluation of integrals.

We give a few examples.

Example. a) Compute
\[ \int_{0}^{\pi} \sin(x) \, dx. \]

b) Compute
\[ \int_{1}^{4} \frac{2x^2 + 1}{\sqrt{x}} \, dx. \]

Solution. a) Since \( \frac{d}{dx}(-\cos(x)) = \sin(x) \), we have \(-\cos(x)\) is an anti-derivative of \(\sin(x)\). Using the second part of the fundamental theorem of calculus gives,
\[ \int_{0}^{\pi} \sin(x) \, dx = -\cos(x) \bigg|_{x=0}^{\pi} = 2. \]

b) We first find an anti-derivative. As the indefinite integral is linear, we write
\[ \int \frac{2x^2 + 1}{\sqrt{x}} \, dx = \int 2x^{3/2} + x^{-1/2} \, dx = 2 \int x^{3/2} \, dx + \int x^{-1/2} \, dx = \frac{4}{5} x^{5/2} + 2 x^{1/2} + C. \]

With this anti-derivative, we may then use FTC II to find
\[ \int_{1}^{4} \frac{2x^2 + 1}{\sqrt{x}} \, dx = \left. \frac{4}{5} x^{5/2} + 2 x^{1/2} \right|_{x=1}^{4} \]
\[ = \frac{4}{5} 4^{5/2} + 24^{1/2} - \left( \frac{4}{5} + 2 \right) \]
\[ = 128/5 + 20/5 - (4/5 + 10/5) \]
\[ = 134/5. \]

\[ \square \]
1.3 Differentiating integrals.

FTC I plays an important role in the proof of FTC II. It is also used to find the derivatives of integrals.

Example. Find
\[
\frac{d}{dx} \int_0^x \sin(t^2) \, dt \quad \frac{d}{dx} \int_{x^2}^x \sin(t^2) \, dt \quad \frac{d}{dx} \int_1^x \frac{1}{t} \, dt
\]

Is the function \( L(x) = \int_1^x \frac{1}{t} \, dt \) increasing or decreasing? Is the graph of \( L \) concave up or concave down?

1.4 The net change theorem

Since \( F \) is always an anti-derivative of \( F' \), one consequence of part II of the fundamental theorem of calculus is that if \( F'' \) is continuous on the interval \([a, b]\), then
\[
\int_a^b F'(t) \, dt = F(b) - F(a).
\]

This helps us to understand some common physical interpretations of the integral.

For example, if \( p(t) \) denotes the position of an object. More precisely, if an object is moving along a line and \( p \) gives the number of meters the object lies to the right of a reference point, then \( p' = v \) is the velocity of the object. The definite integral
\[
p(b) - p(a) = \int_a^b v(t) \, dt
\]

denotes the net change in position of the object during the interval \([a, b]\). Note that if \( v \) is measured in meters/second, then units on \( v(t) \) \( dt \) would be meters/second \( \times \) seconds so the equation (1) is a sophisticated version of the familiar fact that distance = rate \( \times \) time.

To give a less familiar example, suppose we have a rope whose thickness varies along its length. Fix one end of the rope to measure from and let \( m(x) \) denote the mass in kilograms of the rope from 0 to \( x \) meters along the rope. If we take the derivative, \( \frac{dm}{dx} = \lim_{h \to 0} m(x + h) - m(x)/h \), then this represents an average mass of the rope near \( x \) whose units are kilograms/meter. If we integrate this linear density and observe that \( m(0) = 0 \), then we recover the mass
\[
m(x) = \int_0^x \frac{dm}{dx} \, dx.
\]

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