1 The substitution rule.

- Recall the chain rule and restate as the substitution rule.
- u-substitution, bookkeeping for integrals.
- Definite integrals, changing limits.
- Symmetry-integrating even and odd functions.

1.1 The substitution rule.

Recall the chain rule: If $F' = f$ and g is differentiable, then

$$(F \circ g)'(x) = F'(g(x))g'(x).$$

We can restate this as:

The substitution rule. If F is an anti-derivative of f and g is a differentiable function, then $F \circ g(x)$ is an anti-derivative of $(f \circ g)(x)g'(x)$. In other words,

$$F \circ g(x) = \int f(g(x))g'(x)\,dx.$$

1.2 u-substitution

The following procedure is very useful way to use the substitution rule.

To evaluate an integral

$$\int f(g(x))g'(x)\,dx$$

set $u = g(x)$ and then $du = g'(x)dx$ making these substitutions gives

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) = F(g(x)) + C.$$

A simple example might be clearer.

Example. Find

$$\int 2x\sin(x^2)\,dx.$$

Solution. Set $u = x^2$ and then $du = 2xdx$. Making these substitutions gives

$$\int 2x\sin(x^2)\,dx = \int \sin u\,du = \cos u + C = \cos(x^2) + C.$$

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Exercise. Check our answer by differentiating.

Below is a slightly more interesting example. In this example, we do not find exactly the derivative of $u = g(x)$ hiding in the integral. However, if we multiply by appropriate constants, we can still use this method.

Example. Find

$$\int \frac{1}{(1-2x)^2}\, dx.$$

Solution. In this example, we only need to substitute by the linear function $u = 1 - 2x$ and then $du = (-2)dx$. In this case, we need to divide by -2 to obtain $\frac{-1}{2}du = dx$. Then we obtain,

$$\int \frac{1}{(1-2x)^2}\, dx = \frac{-1}{2}\int \frac{1}{u^2}\, du = \frac{1}{2}u^{-1} = \frac{1}{2}\frac{1}{1-2x} + C.$$

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Exercise.

$$\int \sin x \cos x \, dx$$

1.3 Definite integrals.

If we use the substitution rule in a definite integral, we obtain

$$\int_a^b f(g(x))g'(x)\, dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\, du.$$

Sometimes, it is more convenient to change the limits of integration, rather than to replace u by $g(x)$ after integrating.

We give a simple example of how this works.

Example. Find

$$\int_1^4 \sqrt{2x+1}\, dx.$$

Solution. Set $u = 2x + 1$ and then $du = 2dx$. If $x = 1$, then $u = 3$ and if $x = 4$, then $u = 9$. Thus,

$$\begin{aligned} \int_1^4 \sqrt{2x+1}\, dx &= \frac{1}{2}\int_3^9 u^{1/2}\, du \\ &= \frac{1}{2}\frac{2}{3}\, u^{2/3}\Big|_3^9 \\ &= \frac{1}{3}(9^{3/2} - 3^{3/2}) \end{aligned}$$

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1.4 symmetry

The substitution $u = -x$ gives

$$\int_0^a f(x)\,dx = \int_{-a}^0 f(-u)\,du.$$

If f is odd, or even, this simplifies further.

For even functions,

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$

For odd funcitons,

$$\int_{-a}^a f(x)\,dx = 0.$$

Example. Find

$$\int_{-2}^2 x^3 + x^2 + x + 2\,dx \qquad \int_{-1}^1 x^{101}\sin(x^{100})\,dx \qquad \int_{-10}^{11} x\,dx.$$

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