1 Lecture 12: The derivative.

1.1 Outline

- Definition of the derivative at a point and differentiability
- Derivatives of linear and constant functions
- Definition of a tangent line
- Example: finding a tangent line
- Example: a function which is not differentiable
- Example: Derivative of x^3

1.2 The derivative

Definition. Given a function f, which is defined in an interval containing a, we may define the *derivative* of f at a by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exists. If the $f'(a)$ exists, we say that f is *differentiable* at a.

The expression $\frac{f(x)-f(a)}{x-a}$ gives an average rate of change and the limit defines the rate of change of f at a point a. We call the expression $\frac{f(x)-f(a)}{x-a}$ the *difference quotient* of f at a.

An equivalent definition that is sometimes useful is

$$f'(a) = \lim_{h \to} \frac{f(a+h) - f(a)}{h}.$$

You should try both formulations and learn which is easier for you to use. We will see problems where it is helpful to know both formulations.

We find the derivative of some simple functions.

1.3 Example

Example. Find the derivative of a constant function, $f(x) = c$.

Solution. If f is a constant then the difference quotient

$$\frac{f(x) - f(a)}{x - a} = \frac{c - c}{x - a} = 0, \qquad x \neq a.$$

The limit of the difference quotient is zero. Thus $f'(x) = 0$. ∎

Example. Find the derivative of the linear function $g(x) = x$, then the difference quotient

$$\frac{g(x) - g(a)}{x - a} = 1.$$

Solution. We take the limit of the difference quotient to find

$$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \lim_{x \to a} 1 = 1.$$

For this function $g'(a) = 1$. ∎

Example. Find the derivative of $f(x) = 1/x$ at 2.

Definition. The *tangent line* to the graph of a function f at a point a is the line which passes through $(a, f(a))$ and has slope the derivative of f at a, $f'(a)$. In point slope form the equation of the tangent line is

$$y - f(a) = f'(a)(x - a).$$

Example. Find the tangent line to the parabola $y = x^2$ at $x = 2$. Check your answer by graphing the parabola and the tangent line.

Solution. We first need to find the derivative of $f(x) = x^2$ at 2. Using the definition, this is

$$\lim_{h \to 0} \frac{(2 + h)^2 - 2^2}{h} = \lim_{h \to 2} 4 + h = 4.$$

Thus the tangent line passes through (2,4) and has slope 4. The equation is

$$y - 4 = 4(x - 2).$$

Simplifying to put this in point-slope form gives

$$y = 4x - 4.$$

Be sure to check. ∎

Example. Can you find the derivative of $f(x) = |x|$ at $x = 0$? At other values of x?

Solution. If we write down the difference quotient we find that

$$\frac{f(x) - f(0)}{x} = \frac{|x|}{x}.$$

We have computed the one-sided limits for this function and we know

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1 \qquad \lim_{x \to 0^-} \frac{|x|}{x} = -1.$$

Since the two one-sided limits have different values, the limit of the difference quotient at 0 does not exist. The function $f(x) = |x|$ is not differentiable at 0. ∎

Example. Find the derivative of $f(x) = x^3$ at a general point b.

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