1 Lecture 13: The derivative as a function.

1.1 Outline

- Definition of the derivative as a function. definitions of differentiability.
- Power rule, derivative the exponential function
- Derivative of a sum and a multiple
- Differentiability implies continuity.
- Example: Finding a derivative.

1.2 The derivative

Definition. Given a function f, we may define a new function f', which we call *the derivative of f* by the rule that $f'(x)$ is the derivative at x.

Recalling the definition of the derivative at a point, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. The domain of f' is exactly the set of points where f is differentiable.

Example. Let f be a function with domain $[0,1]$ and suppose that $f(x) = 0$ for all x in the domain $[0,1]$. What is the derivative of f?

Solution. As you might guess there is a subtle point here. Of course the derivative f' is 0 when it is defined. If we go back and re-examine the definition of the limit, we see that to take the limit of the difference quotient $\frac{f(x+h)-f(x)}{h}$, we need the difference quotient to be defined for h in an interval that contains 0 (though not at 0, of course). If $x = 0$, the difference quotient is not defined for any negative h. This means we cannot take the limit from the left at 0. Similarly, we cannot take the limit from the right at 1. Thus the derivative does not exist at 0 or 1.

The answer is

$$f'(x) = 0, \qquad \text{for } x \text{ in the domain } (0,1).$$

In general, a function defined on a closed interval is not differentiable at the endpoints of the interval. This point will come up again–but we do not need to worry about it too much now. ∎

An equivalent definition that is sometimes useful is

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}.$$

We will use both forms of the definition of the derivative.

We will sometimes use a different notation for the derivative, d/dx. The symbol f' and the Leibniz notation df/dx both denote the same function,

$$\frac{df}{dx} = f'.$$

The Leibniz notation is particular convenient for functions that are given by a formula but have no name. For example, in the next paragraph, we find the derivative of x^n,

$$\frac{d}{dx} x^n.$$

1.3 Some formulae

We have two important differentiation formulas:

$$\frac{d}{dx} x^n = n x^{n-1}, n = 1, 2, 3, \ldots$$

and

$$\frac{d}{dx} e^x = e^x.$$

The first can be proved by an algebraic simplification:

$$\frac{y^n - x^n}{x - y} = \frac{(x - y)(x^{n-1} + x^{n-2} y + \ldots + x y^{n-2} + y^{n-1})}{x - y}.$$

Since the factor $x - y$ cancels, we have

$$\frac{d}{dx} x^n = \lim_{y \to x} \frac{y^n - x^n}{x - y} = \lim_{y \to x} (x^{n-1} + x^{n-2} y + \ldots + x y^{n-2} + y^{n-1}) = n x^{n-1}.$$

In the limit each of the terms $x^{n-1-k} y^k$ converges to x^{n-1} and we can see that there n terms altogether.

Computing the second derivative is more difficult. Let b^x be an exponential function to an arbitrary base, $b > 0$. From the properties of b^x, we have

$$\frac{b^{x+h} - b^x}{h} = \frac{b^x b^h - b^x}{h} = \frac{b^h - 1}{h} b^x.$$

It is true that the limit $\lim_{h \to 0} \frac{b^h - 1}{h} = m(b)$ exists. We will assume this fact. The number e is special because it is the only number where this limit is 1,

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1. \tag{1}$$

The property (1) can be used to define e. Thus we have that the function e^x is its own derivative,

$$\frac{d}{dx}e^x = e^x.$$

1.4 Derivatives of sums

Theorem 1 *If f and g are differentiable at x and c is a real number, then $f+g$ and cf are differentiable at x and*

$$(f+g)'(x) = f'(x) + g'(x) \qquad \textit{and} \qquad (cf)'(x) = cf'(x).$$

Proof. We consider the difference quotient for $f+g$ and write as

$$\frac{(f+g)(y)-(f+g)(x)}{x-y} = \frac{f(y)-f(x)}{x-y} + \frac{g(y)-g(x)}{x-y}.$$

Since we know each of the difference quotients on the right has a limit, we may use the sum rule for limits

$$\lim_{y\to x}\frac{(f+g)(y)-(f+g)(x)}{x-y} = \lim_{y\to x}\frac{f(y)-f(x)}{x-y} + \lim_{y\to x}\frac{g(y)-g(x)}{x-y}.$$

Thus $(f+g)'(x) = f'(x) + g'(x)$.

We omit the proof of the second one. ∎

Example. Find the derivative of $f(x) = 3x^4 + 4x^3$.

Solution. $12(x^3 + x^2)$. ∎

1.5 Differentiability and continuity.

Theorem 2 *If f is differentiable at x, then f is continuous at x.*

Proof. To show f is continuous at x, we will show that

$$\lim_{y\to x}(f(y) - f(x)) = 0.$$

We can use the product rule for limits and the differentiability of f to see that

$$\lim_{y\to x}(f(y)-f(x)) = \lim_{y\to x}\frac{f(y)-f(x)}{y-x}(y-x)) = f'(x)\cdot 0 = 0.$$

∎

1.6 Examples

Example. Find the derivative of $f(x) = \sqrt{x}$.

Example. Let $f(x) = 1/x$. Find all values x where the slope of the tangent line at x is 4. Find all values x where the slope of the tangent line is -4.

Find all tangent lines to the graph of f which are parallel to the line $y = -4x$.

Example. Sketch the graph of $\sin(x)$ and make a rough sketch of the graph of the derivative, $\sin'(x)$. Can you guess the derivative of sin?

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