1 Lecture 14: The product and quotient rule

1.1 Outline

- The product rule, the reciprocal rule, and the quotient rule.
- Power rule, derivative the exponential function
- Derivative of a sum
- Differentiability implies continuity.
- Example: Finding a derivative.

1.2 The derivative

Theorem 1 *Suppose that f and g are two functions which are differentiable at a point x, then fg is differentiable at x and*

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Proof. The proof depends on rewriting the difference quotient for fg in terms of the difference quotients for f and g. This depends on the trick of adding and subtracting $f(x)g(x+h)$ as follows

$$\begin{aligned}
\frac{f(x+h)g(x+h) - f(x)g(x)}{h} &= \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \frac{f(x+h) - f(x)}{h} g(x+h) + f(x)\frac{g(x+h) - g(x)}{h}.
\end{aligned}$$

We know that the difference quotients for f and g have a limit as h tends to zero. Since g is differentiable at x, it is continuous and we have

$$\lim_{h \to 0} g(x+h) = g(x).$$

Thus we may use the rules for sums and products of limits to obtain that

$$\begin{aligned}
(fg)'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \lim_{h \to 0} g(x+h) + f(x) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}$$

∎

One way to understand this rule is to think of a rectangle whose length ℓ and width w are given by $\ell(t) = a + bt$ and $w(t) = c + dt$. Then the area will be given by

$$\ell(t)w(t) = (\ell_0 + mt)(w_0 + nt) = \ell_0 w_0 + (mw_0 + \ell_0 n)t + mnt^2.$$

At $t = 0$, the instantaneous rate of change of the area will be the coefficient of t, $mw_0 + \ell_0 n$. Since m is the rate of change of ℓ and n is the rate of change of w, the rate of change of the product is exactly what we see in the product rule.

Example. Compute that derivative $f(x) = x^3 e^x$.

Solution. We use the Leibniz notation,

$$\begin{aligned} \frac{d}{dx}(x^3 e^x) &= (\frac{d}{dx}x^3)e^x + x^3 \frac{d}{dx}e^x \\ &= 3x^2 e^x + x^3 e^x. \end{aligned}$$

∎

Example. Find the derivative of x^6 by writing $x^6 = x^3 \cdot x^3$ and applying the product rule.

Solution. We write $x^6 = x^3 \cdot x^3$ and apply the product rule,

$$\frac{d}{dx}x^6 = \frac{d}{dx}(x^3 \cdot x^3) = x^3 \frac{d}{dx}x^3 + (\frac{d}{dx}x^3)x^3.$$

Computing the derivatives gives

$$3x^2 \cdot x^3 + x^3 \cdot 3x^2 = 6x^5.$$

∎

1.3 Reciprocals

We find the derivative of a reciprocal or the multiplicative inverse of a function.

Theorem 2 *If g is differentiable at x and $g(x) \neq 0$, then $1/g$ is differentiable at x and we have*

$$\left(\frac{1}{g}\right)'(x) = \frac{-g'(x)}{g(x)^2}.$$

Proof. We write out the difference quotient for $1/g$, obtain a common denominator and simplify to express it in terms of the difference quotient for g,

$$\begin{aligned}\frac{1}{h}\left(\frac{1}{g(x+h)}-\frac{1}{g(x)}\right) &= \frac{1}{h}\frac{g(x)}{g(x)g(x+h)}-\frac{g(x+h)}{g(x)g(x+h)}\\ &= \frac{-1}{h}(g(x+h)-g(x))\frac{1}{g(x+h)g(x)}.\end{aligned}$$

Now we may use the limit laws and that $1/g$ is continuous at x to write

$$\begin{aligned}\left(\frac{1}{g}\right)'(x) &= -\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\cdot\lim_{h\to 0}\frac{1}{g(x)g(x+h)}\\ &= \frac{-g'(x)}{g(x)^2}.\end{aligned}$$

∎

Example. Use this rule to find the derivative

$$\frac{d}{dx}\frac{1}{x^4}.$$

Solution. $\frac{d}{dx}\frac{1}{x^4} = \frac{-4x^3}{(x^4)^2} = \frac{-4}{x^5}$. ∎

We can use the reciprocal rule to extend the power rule to negative exponents.

Example. Use the reciprocal rule to find the derivative

$$\frac{d}{dx}x^{-n}, \qquad \text{for } n = 1, 2, 3, \ldots.$$

Solution. $\frac{d}{dx}\frac{1}{x^n} = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1}$. ∎

1.4 Quotient rule

Finally we give the quotient rule. Note that it is often simpler to rewrite a quotient as a product and avoid the quotient rule.

Theorem 3 *If f and g are differentiable at x and $g(x) \neq 0$, then f/g is differentiable at x and*

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}.$$

Proof. We may prove this writing $f/g = f \cdot \frac{1}{g}$ and using the product and the reciprocal rule.

$$\left(\frac{f}{g}\right)'(x) = (f\frac{1}{g})'(x) = f'(x)\frac{1}{g(x)} + f(x)\frac{-g'(x)}{g(x)^2}$$

We may simplify this last expression by obtaining a common denominator.

$$f'(x)\frac{1}{g(x)} + f(x)\frac{-g'(x)}{g(x)^2} = f'(x)\frac{g(x)}{g(x)^2} + f(x)\frac{-g'(x)}{g(x)^2} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

∎

Example. Find the tangent line to

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

at $x = 1$.

Solution. The tangent line will pass through the point $(1, f(1)) = (1, 0)$. We need the derivative of f to compute the slope. We use the quotient rule to find the derivative of f,

$$\begin{aligned} f'(x) &= \frac{(\frac{d}{dx}(x^2 - 1))(x^2 + 1) - (x^2 - 1)\frac{d}{dx}(x^2 + 1)}{(x^2 + 1)^2} \\ &= \frac{2x(x^2 + 1) - (x^2 - 1)2x}{(x^2 + 1)^2} \\ &= \frac{4x}{(x^2 + 1)^2}. \end{aligned}$$

At 1, we have $f'(1) = 1$. Thus the tangent line has the equation

$$y - 0 = 1(x - 1).$$

We simplify this to give $y = x - 1$ as the equation of the tangent line. ∎