1 Lecture 15: Rates of change, higher derivatives

1.1 Outline

- Rates of change, velocity, acceleration.
- Interpreting position and velocity graphs
- Equation of motion for objects falling under constant gravity
- Higher-order derivatives

1.2 Rates of change

If $f(t)$ is a function depending on time, then we can write down the average rate of change on an interval $[t_1, t_2]$,

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}.$$

We have defined the instantaneous rate of change at a time t as the limit

$$f'(t) = \lim_{r \to t} \frac{f(r) - f(t)}{r - t}.$$

This is also called the derivative. Thus, instantaneous rate of change is another name for the derivative. In applications the name rate of change is more descriptive.

If $s(t)$ gives the position of a particle as it moves along a line, then

$$v(t) = \lim_{k \to 0} \frac{s(t+k) - s(t)}{k}$$

gives the instantaneous velocity. If position is measured in meters and time in seconds, the velocity v is measured in meters/second. We will abbreviate meters by m and seconds by s so that the units for velocity are m/s.

As a second example, let $v(t)$ be a function which gives the velocity of a particle, measured in meters per second, at time t seconds. Then the instantaneous rate of change of velocity is

$$a(t) = \lim_{r \to t} \frac{v(r) - v(t)}{r - t}.$$

This is called *acceleration* and if velocity is measured in meters per second (or m/s) and time is measured in seconds, then the units for acceleration will be

$$\frac{\mathrm{m/s}}{\mathrm{s}} = \mathrm{m/s}^2.$$

If $a > 0$, then the velocity is increasing and if $a < 0$, then velocity is decreasing.

We now give a simple, but very important equation for the height of an object moving under the influence of gravity. At the moment, we can only write down the equation. By the end of the semester, we will know how to recover h from a.

Example. If an object is thrown near the surface of the Earth, its height at time t units after being thrown is given by

$$h(t) = -\frac{1}{2}gt^2 + v_0 t + h_0.$$

Show that the acceleration is the constant $-g$ and find the velocity and height at time $t = 0$.

Solution. Differentiating twice, we have

$$v(t) = v_0 - gt \qquad \text{and} \qquad a(t) = -g.$$

Thus the acceleration is constant. If we set $t = 0$ we find $v(0) = v_0$ and $h(0) = h_0$. ∎

If we measure height in meters, then $g = 9.8$ m/s^2. Note that the negative sign indicates that gravity is pulling the particle down or in the negative direction.

Example. Suppose that we throw a ball up in the air and it returns to ground level after 4 seconds. a) What is the initial velocity? b) What is the greatest height?

Solution. We may assume that the initial height is zero, $h_0 = 0$, and we know that $h(4) = -\frac{1}{2}g4^2 + v_0 4 = 0$. Solving this equation for v_0 gives

$$v_0 = \frac{1}{2}g \cdot 4.$$

When $g = 9.8$ m/s^2, this gives the numerical value $v_0 = \frac{1}{2} \cdot 9.8\text{m/s}^2 \cdot 4\text{s} = 19.6$ m/s.

We know the maximum height will occur when the velocity is zero. Solving $v(t) = v_0 - gt = 0$ gives the maximum occurs when $t = t_{max} = v_0/g$. The maximum height will be $h(t_{max}) = -\frac{1}{2}g(v_0/g)^2 + v_0 v_0/g = \frac{1}{2}v_0^2/g$. Since $v_0 = 19.6\,\text{m/s}$ and $g = 9.8\,\text{m/s}^2$ the maximum height is 19.6 m.

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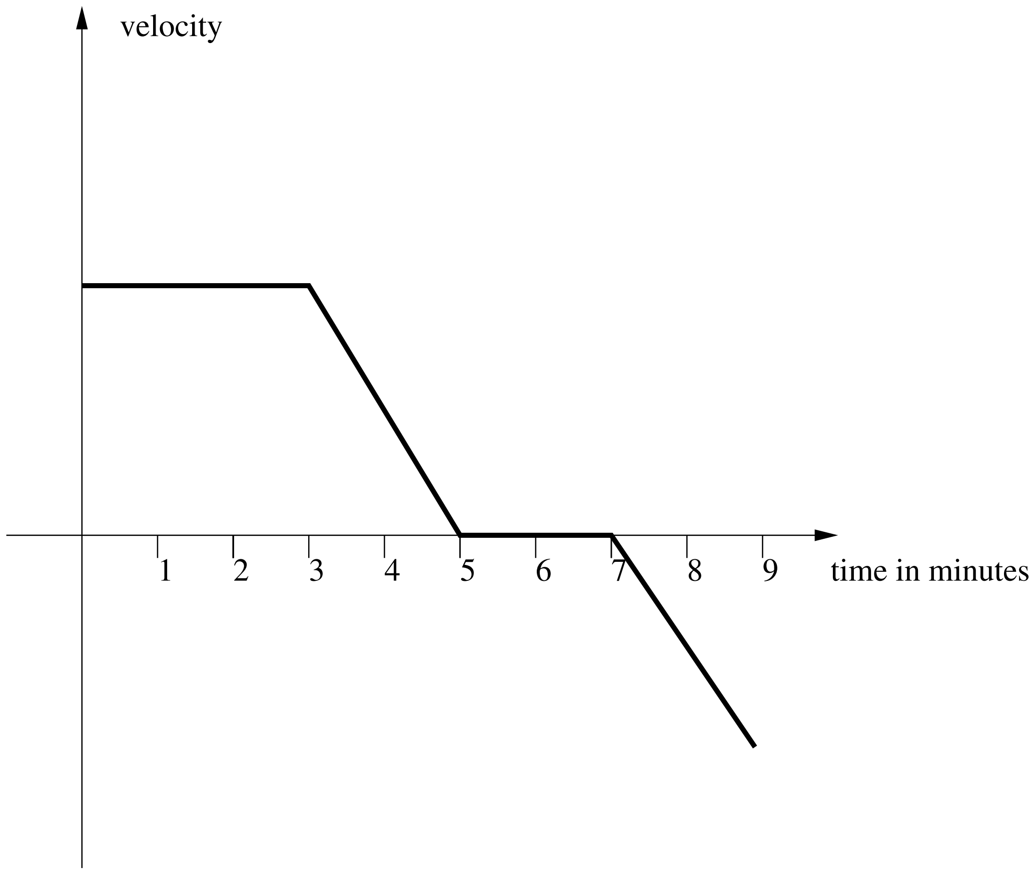
1.3 Interpreting a velocity graph

Example. The graph below gives the velocity of Radar as he walks along the number line. Use the graph below to answer the following questions: a) When is the Radar resting? b) When is Radar moving to the right? c) When is Radar's velocity increasing? d) What information do you need to sketch a graph of Radar's position?



Solution. a) The velocity is zero between 5 and 7 minutes. b) The velocity is positive from 0 to 5 minutes. c) Never. d) A starting point. We might also need units for the vertical axis. ∎

1.4 Higher order derivatives

The acceleration is the derivative of the derivative of position. We call this a second derivative. We often write f'' for the derivative of f' and we call f'' the second derivative. We can define derivatives of any order by

$$f^{(0)} = f \qquad \text{and} \qquad f^{(n)} = f^{(n-1)'}$$

for the result of differentiating n-times. We will also write with the Leibniz notation that

$$f^{(n)} = \frac{d^n f}{dx^n}.$$

Be careful to use the parentheses which help us to distinguish the nth-derivative $f^{(n)}$ from the nth power f^n.

Example. Find

$$\frac{d^2}{dx^2}(x^3 + 1).$$

If f is defined by $f(x) = e^x$, find $f^{(2012)}$.

Solution. The first derivative

$$\frac{d}{dx}(x^3 + 1) = 3x^2.$$

Differentiating again gives

$$\frac{d}{dx}3x^2 = 6x.$$

After spending all day computing 2012 derivatives, we find that $f^{(2012)}(x) = f^{(2011)}(x) = \ldots = f(x) = e^x$.

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Example. What is

$$\frac{d^n}{dx^n}x^n?$$

Solution. Try a few examples and look for a pattern. ∎