

# 1 Lecture 16: Derivatives of trigonometric functions

## 1.1 Outline

- Preliminaries
- The derivatives of  $\sin(t)$  and  $\cos(t)$ .
- The remaining derivatives.

## 1.2 Preliminaries

We recall that the functions  $\sin$  and  $\cos$  are continuous and also the basic limits

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x} = 0.$$

We will also need the addition formula for  $\sin$  and  $\cos$ .

$$\begin{aligned}\sin(x + y) &= \sin(x) \cos(y) + \cos(x) \sin(y) \\ \cos(x + y) &= \cos(x) \cos(y) - \sin(x) \sin(y)\end{aligned}$$

## 1.3 The derivatives of $\sin(x)$ and $\cos(x)$

Our first goal is to find the derivatives of  $\sin$  and  $\cos$ . We have

$$\frac{d}{dx} \sin(x) = \cos(x) \quad \text{and} \quad \frac{d}{dx} \cos(x) = -\sin(x).$$

To establish the formula for the derivative of  $\sin$ , we write the difference quotient and use the addition formula for  $\sin$  to find

$$\begin{aligned}\frac{\sin(x + h) - \sin(x)}{h} &= \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h} \\ &= \sin(x) \frac{\cos(h) - 1}{h} + \cos(x) \frac{\sin(h)}{h}\end{aligned}$$

Using our basic trig limits and the rules for sums and products of limits, we obtain

$$\begin{aligned}\frac{d}{dx} \sin(x) &= \sin(x) \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\ &= \sin(x) \cdot 0 + \cos(x) \cdot 1 = \cos(x).\end{aligned}$$

*Exercise.* Carry out a similar computation to find the derivative of  $\cos(x)$ .

*Example.* Find the tangent line to  $f(x) = \sin(x) \cos(x)$  at  $x = 0$ .

*Solution.* The tangent line passes through the point  $(0, f(0)) = (0, 0)$  and has derivative  $f'(0)$  and has the equation  $y - f(0) = f'(0)(x - 0)$ .

We compute  $f'(x)$  using the product rule,

$$f'(x) = \sin'(x) \cos(x) + \sin(x) \cos'(x) = \cos^2(x) - \sin^2(x).$$

Substituting  $x = 0$  gives  $f'(0) = 1$ . Using that  $(0, f(0)) = (0, 0)$  and  $f'(0) = 1$  gives the tangent line is

$$y = x.$$

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*Example.* When does the graph of the function  $f(x) = x + 2\sin(x)$  have a horizontal tangent line?

*Solution.* The graph will have a horizontal tangent line at values  $x$  which satisfy  $f'(x) = 0$ . We compute  $f'(x) = 1 + 2\cos(x)$ . We will have  $f'(x) = 0$  if  $\cos(x) = -1/2$ . The solutions of this equation in the interval  $[0, 2\pi]$  are  $x = 2\pi/3$  and  $x = 4\pi/3$ . To obtain all solutions, we add an arbitrary multiple of  $2\pi$ . Thus solutions are

$$x = 2\pi/3 + 2k\pi, \quad 4\pi/3 + 2k\pi, \quad k = 0, \pm 1, \pm 2, \dots$$

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## 1.4 Derivatives of the remaining trigonometric functions

In this section, we find the derivatives of the remaining trigonometric functions. To find the derivatives we express the function in terms of  $\sin$  and  $\cos$  and then using the quotient or reciprocal rule.

*Example.* Find the derivative of  $\tan(x)$ .

*Solution.* We recall that  $\tan(x) = \frac{\sin(x)}{\cos(x)}$ . Using the quotient rule, we have

$$\begin{aligned} \frac{d}{dx} \frac{\sin(x)}{\cos(x)} &= \frac{\cos(x) \frac{d}{dx} \sin(x) - \sin(x) \frac{d}{dx} \cos(x)}{\cos^2(x)} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= 1/\cos^2(x) = \sec^2(x). \end{aligned}$$

Thus, we have

$$\frac{d}{dx} \tan(x) = \sec^2(x).$$

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*Exercise.* Establish the differentiation formulae:

$$\frac{d}{dx} \sec(x) = \sec(x) \tan(x) \quad \frac{d}{dx} \cot(x) = -\sec^2(x) \quad \frac{d}{dx} \csc(x) = -\csc(x) \cot(x).$$

We close with a couple of examples:

*Example.* Find the derivative  $f'(0)$  if  $f$  is defined by

$$f(x) = \frac{\cos(x)}{2 + \sin(x)}.$$

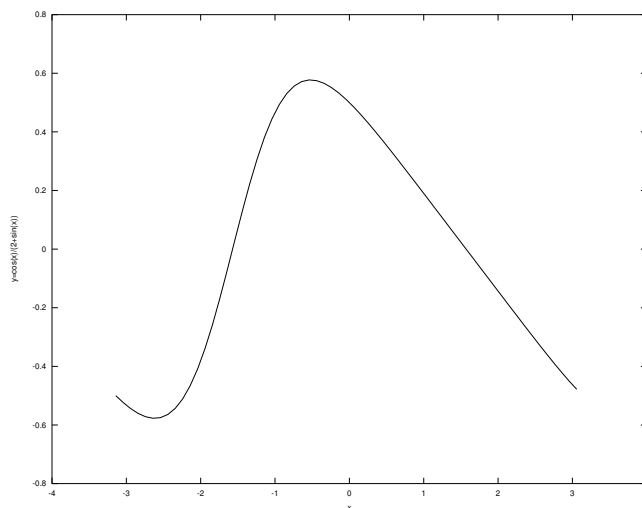
Check your answer by graphing the function and estimating the rate of change at 0.

*Solution.* We use the quotient rule to find the derivative:

$$\begin{aligned} \frac{d}{dx} \frac{\cos(x)}{2 + \sin(x)} &= \frac{(2 + \sin(x)) \frac{d}{dx} \cos(x) - \cos(x) \frac{d}{dx} (2 + \sin(x))}{(2 + \sin(x))^2} \\ &= \frac{(2 + \sin(x))(-\sin(x)) - \cos(x) \cos(x)}{(2 + \sin(x))^2} \\ &= \frac{-2\sin(x) - 1}{(2 + \sin(x))^2} \end{aligned}$$

Thus when  $x = 0$ ,  $f'(0) = -1/4$ .

The graph below suggests this is a reasonable value.



*Example.* If  $f(x) = \sin(x) + \cos(x)$ , find the derivative  $f^{(401)}(x)$ .

Can you find a function  $f$  which satisfies

$$f'' + f = 0?$$

Can each member of the class find a different, correct answer?

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