1 The fundamental theorems of calculus.

- The fundamental theorems of calculus.
- Evaluating definite integrals.
- The indefinite integral-a new name for anti-derivative.
- Differentiating integrals.

Today we provide the connection between the two main ideas of the course. The integral and the derivative.

Theorem 1 *Suppose f is a continuous function on $[a, b]$.*

(FTC I) If F is an anti-derivative of f, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

(FTC II) If $g(x) = \int_a^x f(t)\,dt$, then $g' = f$.

Example. Compute

$$\frac{d}{dx}\int_1^x \frac{1}{t}\,dt.$$

Compute

$$\int_0^3 x^3\,dx.$$

Proof. An idea of the proofs.

FTC I: We let F be an anti-derivative of f and let $P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$. We will express the change of F, $F(b) - F(a)$, as a Riemann sum on the partition. Letting the size of the largest interval in the partition tend to zero, we obtain the integral is equal to the change in F.

We begin by writing

$$F(b)-F(a) = F(x_n)-F(x_{n-1})+F(x_{n-1})-\ldots+F(x_i)-F(x_{i-1})+\ldots+F(x_1)-F(x_0).$$

We recall that F is an anti-derivative of f and apply the mean value theorem on each interval $[x_{i-1}, x_i]$ and find a value c_i so that $F(x_i)-F(x_{i-1}) = f(c_i)(x_i - x_{i-1})$. Thus, we have

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}).$$

Since the right-hand side is a Riemann sum for the integral, we may let the width of the largest subinterval tend to zero and obtain

$$F(b) - F(a) = \int_a^b f(s)\,ds.$$

FTC II:

Write

$$\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

We will show

$$\lim_{h\to 0^+} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x).$$

The reader should write out a similar argument for the limit from the below.

If f is continuous, then f has maximum and minimum values M_h and m_h on the interval $[x, x+h]$. Using the order property of the integral,

$$m_h \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le M_h.$$

As h tends to 0, we have $\lim_{h\to 0^+} M_h = \lim_{h\to 0^+} m_h = f(x)$ since f is continuous. It follows that

$$\lim_{h\to 0^+} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x).$$

∎

1.1 Indefinite integrals.

We use the symbol

$$\int f(x)\,dx$$

to denote the indefinite integral or anti-derivative of f.

The indefinite integral is a function. The definite integral is a number. According FTC I, we can find the (numerical) value of a definite integral by evaluating the indefinite integral at the endpoints of the integral. Since this procedure happens so often, we have a special notation for this evaluation.

$$F(x)|_{x=a}^{b} = F(b) - F(a).$$

Example. Find

$$xa|_{x=a}^{b} \qquad \text{and} \qquad xa|_{a=x}^{y}$$

Solution.

$$ba - a^2 \qquad xy - x^2$$

∎

According to FTC II, anti-derivatives exist provided f is continuous.

The box on page 351 should be memorized. (In fact, you should already have memorized this information when we studied derivatives in Chapter 3 and when we studied anti-derivatives in Chapter 4.)

Example. Verify

$$\int x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2).$$

Solution. According to the definition of anti-derivative, we need to see if

$$\frac{d}{dx}\frac{1}{2}\sin(x^2) = x\cos(x^2).$$

This holds, by the chain rule. ∎

1.2 Computing integrals.

The main use of FTC I is to simplify the evaluation of integrals.

We give a few examples.

Example. a) Compute

$$\int_0^\pi \sin(x)\,dx.$$

b) Compute

$$\int_1^4 \frac{2x^2+1}{\sqrt{x}}\,dx.$$

Solution. a) Since $\frac{d}{dx}(-\cos(x)) = \sin(x)$, we have $-\cos(x)$ is an anti-derivative of $\sin(x)$. Using the second part of the fundamental theorem of calculus gives,

$$\int_0^\pi \sin(x)\,dx = -\cos(x)|_{x=0}^{\pi} = 2.$$

b) We first find an anti-derivative. As the indefinite integral is linear, we write

$$\int \frac{2x^2+1}{\sqrt{x}}\,dx = \int 2x^{3/2} + x^{-1/2}\,dx \quad = \quad 2\int x^{3/2}\,dx + \int x^{-1/2}\,dx = \frac{4}{5}x^{5/2} + 2x^{1/2} + C.$$

With this anti-derivative, we may then use FTC I to find

$$\begin{aligned}
\int_1^4 \frac{2x^2+1}{\sqrt{x}}\,dx &= \left.\frac{4}{5}x^{5/2} + 2x^{1/2}\right|_{x=1}^{4} \\
&= \frac{4}{5}4^{5/2} + 24^{1/2} - (\frac{4}{5} + 2) \\
&= 128/5 + 20/5 - (4/5 + 10/5) \\
&= 134/5.
\end{aligned}$$

∎

Example. Find

$$\int_0^{\sqrt{\pi}} 2x\cos(x^2)\,dx.$$

Solution. We recognize that $\sin(x^2)$ is an anti-derivative of $2x\cos(x^2)$,

$$\int 2x\cos(x^2)\,dx = \sin(x^2) + C.$$

Thus,

$$\int_0^{\sqrt{\pi}} 2x\cos(x^2)\,dx = \sin(x^2)\Big|_{x=0}^{\sqrt{\pi}} = 0 - 0.$$

∎

Here, is a more involved example that illustrates the progress we have made.

Example. Find

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \sin(k/n).$$

Solution. We recognize that

$$\frac{1}{n}\sum_{k=1}^{n} \sin(k/n)$$

is a Riemann sum for an integral. The points x_k, $k = 0, \dots, n$ divide the interval $[0, 1]$ into n equal sub-intervals of length $1/n$. Thus, we may write the limit as an integral

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \sin(k/n) = \int_0^1 \sin(x)\,dx.$$

To evaluate the resulting integral, we use FTC I. An anti-derivative of $\sin(x)$ is $-\cos(x)$, thus

$$\int_0^1 \sin(x)\,dx = -\cos(x)|_{x=0}^{1} = 1 - \cos(1).$$

∎

1.3 Differentiating integrals.

FTC II shows that any continuous function has an anti-derivative and can be used to find the derivatives of integrals.

Example. Find

$$\frac{d}{dx}\int_0^x \sin(t^2)\,dt \qquad \frac{d}{dx}\int_{x^2}^x \sin(t^2)\,dt \qquad \frac{d}{dx}\int_1^x \frac{1}{t}\,dt$$

Is the function $L(x) = \int_1^x \frac{1}{t}\,dt$ increasing or decreasing? Is the graph of L concave up or concave down?

1.4 The net change theorem

Since F is always an anti-derivative of F', one consequence of part II of the fundamental theorem of calculus is that if F' is continuous on the interval $[a, b]$, then

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

This helps us to understand some common physical interpretations of the integral.

For example, if $p(t)$ denotes the position of an object. More precisely, if an object is moving along a line and p gives the number of meters the object lies to the right of a reference point, then $p' = v$ is the velocity of the object. The definite integral

$$p(b) - p(a) = \int_a^b v(t)\,dt \tag{2}$$

denotes the net change in position of the object during the interval $[a, b]$. Note that if v is measured in meters/second, then units on $v(t)dt$ would be meters/second $\times$ seconds so the equation (2) is a sophisticated version of the familiar fact that distance $=$ rate $\times$ time.

To give a less familiar example, suppose we have a rope whose thickness varies along its length. Fix one end of the rope to measure from and let $m(x)$ denote the mass in kilograms of the rope from 0 to x meters along the rope. If we take the derivative, $\frac{dm}{dx} = \lim_{h\to 0} m(x + h) - m(x)h$, then this represents an average mass of the rope near x whose units are kilograms/meter. If we integrate this linear density and observe that $m(0) = 0$, then we recover the mass

$$m(x) = \int_0^x \frac{dm}{dx}\,dx.$$