1 Lecture 36: Net change

- FTC 1 gives the net change of a function.
- Distance travelled and displacement.
- Density and mass

1.1 The net change theorem

Since F is always an anti-derivative of F', one consequence of part I of the fundamental theorem of calculus is that if F' is continuous on the interval $[a, b]$, then

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

If we think about the proof of this theorem which relied on writing the change over the $[a, b]$, as a sum of changes over small intervals, this will help us to understand some common physical interpretations of the integral.

For example, if $p(t)$ denotes the position of an object. More precisely, if an object is moving along a line and p gives the number of meters the object lies to the right of a reference point, then $p' = v$ is the velocity of the object. The definite integral

$$p(b) - p(a) = \int_a^b v(t)\,dt \qquad (1)$$

denotes the net change in position of the object during the interval $[a, b]$. Note that if v is measured in meters/second and time t is measured in seconds, then the units for $v(t)dt$ would be meters/second $\times$ seconds so the equation (1) is a sophisticated version of the familiar fact that distance $=$ rate $\times$ time.

We let $P = \{a = t_0, t_i, \ldots, t_n = b\}$ be a partition for $[a, b]$ and $\{c_1, \ldots, c_n\}$ are sample points, we consider a Riemann sum for the integral $\int_a^b v(t)\,dt$,

$$\sum_{i=1}^{n} v(c_i)(t_i - t_{i-1}). \qquad (2)$$

Each term represents the change in position in the interval $[t_{i-1}, t_i]$. As $\|P\|$ tends to zero, the sum should approach the total displacement in the interval.

If instead of displacement, we want the total distance travelled on the interval $[a, b]$, then we can replace $v(c_i)$ by $|v(c_i)|$ to get the distance travelled on the interval $[t_{i-1}, t_i]$. Summing gives the total distance travelled is approximately

$$\sum_{i=1}^{n} |v(c_i)|(t_i - t_{i-1}).$$

Letting $\|P\|$ tend to zero suggests that the total distance travelled should be

$$\int_a^b |v(t)|\,dt.$$

Example. Suppose the position of an object at time t is $v(t) = 5 - t$. Find the displacement and the total distance travelled on the interval $[0, 7]$.

Solution. To calculate the total displacement or change in position, we consider

$$\int_0^7 (5 - t)\, dt = (5t - \frac{1}{2}t^2)\Big|_{t=0}^{7} = 35 - 49/2 = 21/2.$$

This means that the particle has moved 21/2 units to the right. Note that for part of the interval, the velocity is negative. Thus the total distance travelled is greater than the change in displacement.

To find the total displacement, we consider the integral

$$\int_0^7 |5 - t|\, dt.$$

It is not easy to find the anti-derivative of a function with absolute values. A better approach is to divide the interval $[0, 7]$ into subintervals where $5 - t$ is either non-negative or non-positive. When $5 - t$ is positive or zero, we may replace $|5 - t|$ by $5 - t$ and when $5 - t$ is negative or zero, we may replace $|5 - t|$ by $t - 5$:

$$\int_0^7 |5 - t|\, dt = \int_0^5 (5 - t)\, dt + \int_5^7 (t - 5)\, dt.$$

Using FTCI to evaluate the integrals gives

$$\int_0^5 (5 - t)\, dt = (5t - \frac{t^2}{2})\ |_0^5 = 25 - 25/2 = 25/2.$$

Also

$$\int_5^7 (t - 5)\, dt = (t^2/2 - 5t)\ |_5^7 = (49/2 - 35) - (25/2 - 25) = 12 - 10 = 2.$$

Thus the total distance travelled is 29/2 and this is larger than the change in displacement.

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1.2 Linear density and total mass

To give a less familiar example, suppose we have a rope whose thickness varies along its length. Fix one end of the rope to measure from and let $m(x)$ denote the mass in kilograms of the rope from 0 to x meters along the rope. If we take the derivative, $\frac{dm}{dx} = \lim_{h \to 0} m(x + h) - m(x)h$, then this represents an average mass of the rope near x whose units are kilograms/meter. If we integrate this linear density and observe that $m(0) = 0$, then we recover the mass

$$m(x) = \int_0^x \frac{dm}{dx}\, dx.$$

Example. If a rope has linear density of $1 + 0.1 \sin(0.2x)$ grams per centimeter, what is the mass of one meter of rope.

Solution. The total mass of one meter or 100 centimeters is

$$\begin{aligned}\int_0^{100} 1 + 0.1\sin(0.2x)\,dx &= (x - \frac{0.1}{0.2}\cos(0.2x))\,|_{x=0}^{100} \\ &= 100 - \frac{1}{2}\cos(0.2\cdot 100) - (0 - \frac{1}{2}\cos(0)) \\ &= 100 + \frac{1}{2}(\cos(0) - \cos(20)).\end{aligned}$$

The mass is measured in grams. ∎

Example. If cars pass a point on the highway at a rate of $5+\frac{1}{t}$ cars per minute. Find the total number of course that pass by between $t = 20$ minutes to $t = 50$ minutes.

Solution. The total should be

$$\int_{20}^{50} (5 + 1/t)dt.$$

If we think about a term in a Riemann sum, $(5 + 1/t_i)(t_i - t_{i-1})$, this represents the number of cars per minute times a number of minutes and gives total cars passing in the interval of time $[t_{i-1}, t_i]$.

Evaluating the integral gives

$$\int_{20}^{50} (5 + 1/t)dt = (5t + \ln(t))\,|_{t=20}^{50} = 150 + \ln(5/2).$$

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November 16, 2012