1 Lecture 37: The substitution rule.

- Recall the chain rule and restate as the substitution rule.
- *u*-substitution, bookkeeping for integrals.
- Definite integrals, changing limits.
- Symmetry-integrating even and odd functions.

1.1 The substitution rule.

Recall the chain rule: If F' = f and g is differentiable, then

$$(F \circ g)'(x) = F'(g(x))g'(x).$$

We can restate this as:

The substitution rule. If F is an anti-derivative of f and g is a differentiable function, then $F \circ g(x)$ is an anti-derivative of $(f \circ g)(x)g'(x)$. In other words,

$$F \circ g(x) = \int f(g(x))g'(x) \, dx$$

1.2 *u*-substitution

The Leibniz notation provides a convenient way to keep track of the substitution rule. We let

$$u = g(x), \qquad du = g'(x)dx. \tag{1}$$

To evaluate the indefinite integral

$$\int f(g(x))g'(x)\,dx$$

set u = g(x) and then du = g'(x)dx making these substitutions gives

$$\int f(g(x))g'(x) \, dx = \int f(u) \, du = F(u) = F(g(x)) + C$$

A simple example might be clearer.

Example. Find

$$\int 2x\sin(x^2)\,dx.$$

Solution. Set $u = x^2$ and then du = 2xdx. Making the substitutions as in (1) gives

$$\int 2x \sin(x^2) \, dx = \int \sin u \, du = \cos u + C = \cos(x^2) + C$$

Exercise. Check our answer by differentiating.

Below is a slightly more interesting example. In this example, we do not find exactly the derivative of u = g(x) hiding in the integral. However, we may multiply the equation du = g'(x)dx by a constant and still use this method.

Example. Find

$$\int \frac{1}{(1-2x)^2} \, dx.$$

Solution. In this example, we only need to substitute by the linear function u = 1 - 2x and then du = (-2)dx. In this case, we need to divide by -2 to obtain $\frac{-1}{2}du = dx$. Then we obtain,

$$\int \frac{1}{(1-2x)^2} \, dx = \frac{-1}{2} \int \frac{1}{u^2} \, du = \frac{1}{2}u^{-1} = \frac{1}{2}\frac{1}{1-2x} + C.$$

This works because if u = g(x) and v = cg(x), then we have dv = c du = cg'(x) dxby the constant multiple rule for differentiation. If we multiply u by a function h(x), then we would need to use the product rule to find the derivative of hg and life would be more complicated.

Exercise.

$$\int \sin(x) \, \cos(x) \, dx$$

Solution. If we set $u = \sin(x)$, then $du = \cos(x) dx$ and we have

$$\int \sin(x) \, \cos(x) \, dx = \int u \, dx = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2(x) + C.$$

If we set $u = \cos(x)$, then $du = -\sin(x) dx$ and we have

$$\int \sin(x) \, \cos(x) \, dx = -\int u \, dx = -\frac{1}{2}u^2 + C = -\frac{1}{2}\cos^2(x) + C.$$

Check these answers. Explain why we have found two different answers.

1.3 Definite integrals.

Sometimes, it is more convenient to change the limits of integration, rather than to replace u by g(x) after integrating. Thus we make the substitutions,

$$u = g(x),$$
 $du = g'(x)dx,$ if $x = a, u = g(a),$ if $x = b, u = g(b).$ (2)

If we use the substitution rule (2) in a definite integral, we obtain

$$\int_{a}^{b} f(g(x))g'(x) \, dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u) \, du.$$

We give a simple example of how this works.

Example. Find

$$\int_{1}^{4} \sqrt{2x+1} \, dx.$$

Solution. Set u = 2x + 1 and then du = 2dx. If x = 1, then u = 3 and if x = 4, then u = 9. Thus,

$$\int_{1}^{4} \sqrt{2x+1} \, dx = \frac{1}{2} \int_{3}^{9} u^{1/2} \, du$$
$$= \frac{1}{2} \frac{2}{3} u^{2/3} \Big|_{3}^{9}$$
$$= \frac{1}{3} (9^{3/2} - 3^{3/2}) = 9\sqrt{3}.$$

It is also possible to replace u by g(x) to find an anti-derivative as a function of x and then evaluate with the original limits. Here is a solution following this strategy. Solution. Set u = 2x + 1 and then du = 2dx. If x = 1, then u = 3 and if x = 4, then u = 9. Thus,

$$\int \sqrt{2x+1} \, dx = \frac{1}{2} \int u^{1/2} \, du$$
$$= \frac{1}{2} \frac{2}{3} u^{3/2} + C$$
$$= \frac{1}{3} (2x+1)^{3/2} + C$$

Now that we have the anti-derivative, we may use one of the Fundamental Theorems of Calculus to obtain

$$\int_{1}^{4} \sqrt{2x+1} \, dx = \left. \frac{1}{3} (2x+1)^{3/2} \right|_{1}^{4} = \frac{1}{3} (9^{3/2} - 3^{3/2}) = 9 - \sqrt{3}.$$

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Finally, we give one more example, where a bit more algebra is needed.

Example. Find the anti-derivative

$$\int x\sqrt{2x+1}\,dx.$$

Solution. Again, we substitute u = 2x + 1 and du = 2dx or $dx = frac_{12}du$ but this leaves an x. We solve u = 2x + 1 to express $x = \frac{1}{2}(u - 1)$. Making the substitutions, we have

$$\int x\sqrt{2x+1}\,dx = \int \frac{1}{2}(u-1)u^{1/2}\frac{1}{2}du = \frac{1}{4}\int (u^{3/2}-u^{1/2})\,du.$$

Taking the anti-derivative and then replacing u by 2x + 1 gives

$$\frac{1}{4}\int (u^{3/2} - u^{1/2})\,du = \frac{2}{20}u^{5/2} - \frac{2}{12}u^{3/2} + C.$$

And replacing u by 2x + 1 gives

$$\int x \sqrt{2x+1} \, dx == \frac{1}{10} (2x+1)^{5/2} - \frac{1}{6} (2x+1)^{3/2} + C.$$

1.4 Symmetry

The substitution u = -x gives

$$\int_{0}^{a} f(x) \, dx = \int_{-a}^{0} f(-u) \, du$$

If f is odd, or even, this simplifies further.

A function is even if f(-x) = f(x). For even functions we have

$$\int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx.$$

A function is odd if f(-x) = -f(x) and for odd functions,

$$\int_{-a}^{a} f(x) \, dx = 0.$$

Example. Find

$$\int_{-2}^{2} x^{3} + x^{2} + x + 2 \, dx \qquad \int_{-1}^{1} x^{101} \sin(x^{100}) \, dx \qquad \int_{-10}^{11} x \, dx.$$

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