1 Lecture 38: Further transcendental functions

- The derivatives of arctan and arcsin and the corresponding anti-differentiation formula
- Rescaling
- Integrating $(x+a)/(1+x^2)$ and $(x+a)/\sqrt{1-x^2}$.
- Approximating π

1.1 The derivatives of arctan and arcsin

We recall two differentiation formulae.

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}, \qquad -1 < x < 1, \qquad \frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}.$$

Thus, we have two more useful anti-derivatives.

$$\int \frac{dx}{1+x^2} = \arctan(x) + C, \qquad \int \frac{dx}{\sqrt{1-x^2}} + C.$$

Example. Evaluate the integrals

$$\int_0^1 \frac{dt}{1+t^2}, \qquad \int_0^x \frac{dt}{\sqrt{1-t^2}}.$$

Solution.

$$\int_0^1 \frac{dt}{1+t^2} = \arctan(1) - \arctan(0) = \pi/4.$$

$$\int_0^x \frac{dt}{\sqrt{1-t^2}} = \arcsin(x) - \arcsin(0) = \arcsin(x).$$

∎

Thus, the integral can be used to given an alternate definition of the functions arcsin and arctan. This provides a way to compute values of these functions. We may approximate the integrals by Riemann sums. Next semester we will find even more efficient ways to compute these functions.

Example. Can you provide a definition of $\ln(x)$ using an integral?

Solution.

$$\int_1^x \frac{dt}{t}.$$

∎

1.2 Rescaling

In this section, we will find anti-derivatives for

$$\int \frac{dx}{a^2+b^2x^2} \qquad \text{and} \qquad \int \frac{dx}{\sqrt{a^2-b^2x^2}}.$$

Example. Make the substitution $u = 3x/2$ in the integral

$$\int \frac{dx}{4+9x^2}$$

and find an anti-derivative.

Solution. We have $x = 2u/3$ and thus $dx = \frac{2}{3}du$. Thus

$$\int \frac{dx}{4+9x^2} = \frac{2}{3}\int \frac{du}{4+4u^2} = \frac{1}{6}\int \frac{d}{u}1+u^2 = \frac{1}{6}\arctan(3x/2)+C.$$

We may check our answer,

$$\frac{d}{dx}\frac{1}{6}\arctan(2x/3) = \frac{1}{6}\frac{2}{3}\frac{1}{1+4x^2/9} = \frac{1}{9+4x^2}.$$

∎

Example. What can we substitute for x to convert $\sqrt{2-x^2}$ into $\sqrt{2(1-u^2)}$?

Evaluate

$$\int_0^1 \sqrt{2-x^2}.$$

Solution. We will try a substitution of the form $x = au$ and obtain $2-x^2 = 2-a^2u^2$. If we use the distributive law to factor out 2, we obtain $2(1-\frac{a^2}{2}u^2)$ and we see that we should choose $a^2/2 = 1$ in order to obtain an integral involving $1-u^2$. We choose $a = +\sqrt{2}$ for simplicity.

Let $\sqrt{2}u = x$ and then $\sqrt{2-x^2} = \sqrt{2(1-u^2)} = \sqrt{2}\cdot\sqrt{1-u^2}$. Thus, we have the anti-derivative,

$$\int \frac{dx}{\sqrt{2-x^2}} = \sqrt{2}\int \frac{dx}{\sqrt{2}\sqrt{1-u^2}} = \arcsin(u)+C = \arcsin(x/\sqrt{2})+C.$$

We leave it as an exercise to check.

We evaluate the definite integral as

$$\int_0^1 \frac{dx}{\sqrt{2-x^2}} = \arcsin(x/\sqrt{2})\Big|_{x=0}^1 = \arcsin(1/\sqrt{2})-\arcsin(0) = \pi/4.$$

∎

1.3 More general integrals

Example. Find the anti-derivative

$$\int \frac{x+2}{1+4x^2}\, dx.$$

Solution. If we use the linearity of the indefinite integral, we have

$$\int \frac{x+2}{1+4x^2}\, dx = \int x\, dx 1 + 4x^2 + \int \frac{dx}{1+4x^2}.$$

We evaluate the integrals separately. For the first, we use the substitution $u = 1+4x^2$, $\frac{1}{8}du = xdx$ to obtain

$$\int \frac{dx}{1+4x^2} = \frac{1}{8}\int \frac{du}{u} = \frac{1}{8}\ln|u| + C = \frac{1}{8}\ln(1+4x^2) + C.$$

For the second we use the substitution $u = x/2$, $du = \frac{1}{2}$ and obtain

$$\int \frac{dx}{1+4x^2} = \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\arctan(u) + C = \frac{1}{2}\arctan(2x) + C.$$

Thus, altogether we have

$$\int \frac{x+2}{1+4x^2}\, dx = \frac{1}{8}\ln(1+4x^2) + \frac{1}{2}\arctan(2x) + C.$$

There is no loss of generality in dropping the absolute values in the expression $\ln(1+4x^2)$ since $1+4x^2$ is always positive. ∎

We end with a couple of related examples.

Exercise. Find

$$\int \frac{e^x}{1+e^{2x}}\, dx \qquad \int \frac{x+1}{\sqrt{3-x^2}}\, dx, \qquad \int \frac{x^2}{1+x^2}\, dx.$$

1.4 Approximating π

Example. Use the midpoint approximation with four subintervals to approximate the integral

$$\int_0^1 \frac{dx}{1+x^2}.$$

Use this to find an approximation to π. What is the error?

Solution. The midpoint sum is

$$M_4 = \frac{1}{4}(\frac{1}{1+1/64} + \frac{1}{1+9/64} + \frac{1}{1+25/64} + \frac{1}{1+49/64}) \approx 0.78670.$$

Since the exact value of the integral is $\pi/4$, we may approximate π by $4 \cdot 0.78670 \approx 3.1468$. The absolute error is approximately 0.0052079. ∎

November 27, 2012