1 Lecture 39: Exponential growth and decay

- A model for exponential growth and decay
- Fitting our solution to data, doubling time and half-life
- Example: Population growth
- Example: Carbon dating

1.1 A model for population growth

A simple model for a population is to assume that the number of births and deaths is a fixed fraction of the total population. Thus, y is the total population, b is the birth rate or the fraction of the population that give birth each year, then the total number of births per year is by. Similarly if d is the death rate, then dy is the number of deaths per year. The rate of change of the population y with respect to time is $by - dy$. We can express this using a derivative as

$$\frac{dy}{dt} = ky \tag{1}$$

where the constant $k = b - d$. Since y is positive, we have that y is increasing if $k > 0$ and y is decreasing if $k < 0$. It is easy to see that the family of functions $y(t) = P_0 e^{kt}$ are solutions of (1). Here, P_0 is a constant and each choice of P_0 gives a different solution of (1). In fact, these are the only solutions.

Theorem 2 *If y solves (1) on an interval I, then there is a constant P_0 so that $y(t) = P_0 e^{kt}$ on I.*

Proof. If we want to show $y(t) = P_0 e^{kt}$, then we expect that $e^{-kt} y(t)$ is constant. One way to show a function is constant is to show the derivative is zero. We consider the function $f(t) = e^{-kt} y(t)$ and differentiate f using the product rule,

$$f'(t) = (-k)e^{-kt} y(t) + e^{-kt} y'(t) = -ke^{-kt} y + e^{-kt} ky = 0.$$

We have used (1) for the second inequality. Since $f' = 0$ on an interval, f is constant. If we call the constant P_0, then we have $f(t) = P_0 e^{kt}$. ∎

1.2 Model fitting

If a function y is given by $y(t) = P_0 e^{kt}$ and $k > 0$ we say that y grows exponentially. If $k < 0$, then we say that y decays exponentially. In this case we will often replace k by $-k$ and write $y(t) = P_0 e^{-kt}$. Thus the constant k is positive. Once we know that we have exponential growth and decay, we need two additional bits of data to determine the constant k and the value of P_0. Note we have $y(0) = P_0$ so P_0 is the initial value of y.

Example. Suppose that y obeys (1), $y(1) = 2$ and $y(2) = 5$. Find $y(t)$.

Solution. We know that $y(t) = P_0 e^{kt}$. The given information tells us that

$$P_0 e^k = 2, \qquad P_0 e^{2k} = 5.$$

To find k, we may divide these equations and find

$$\frac{P_0 e^{2k}}{P_0 e^k} = \frac{5}{2}.$$

Taking the natural logarithm of both sides, we find $k \ln(e) = \ln(5/2)$. Thus, $y(t) = P_0 e^{t \ln(5/2)}$. Substiuting $t = 1$, we find $P_0 e^{\ln(5/2)} = 2$ or $P_0 = 4/5$. Summarizing,

$$y(t) = \frac{4}{5} e^{t \ln(5/2)} = \frac{4}{5} \left(\frac{5}{2}\right)^t.$$

∎

One of the important features of exponential growth is the existence of a time T during which the population doubles, i.e. that $y(t+T) = 2y(t)$. To see that this doubling property is independent of t, we consider

$$\frac{y(t+T)}{y(t)} = \frac{P_0 e^{k(t+T)}}{P_0 e^{kt}} = e^{kT}.$$

Thus, to find the doubling time we need to solve $e^{kT} = 2$ for T.

Example. If $f(t) = 100e^{0.3t}$, find the doubling time.
Can you find the tripling time?

Solution. We have

$$\frac{f(t+T)}{f(t)} = e^{3T}.$$

We solve the equation $e^{0.3T} = 2$ to find $T = \ln(2)/0.3$.
The same argument gives that the tripling time is $\ln(3)/0.3$. ∎

In the case of exponential decay, the corresponding notion is half-life. This is the time T so that $y(t+T) = \frac{1}{2} y(t)$.

1.3 Example: Exponential growth

Example. Suppose that a population grows at a rate of 3% per day. If the initial population is 100, when will the population reach 1000. What is the doubling time?

Solution. Let $y(t)$ denote the population at time t where t is measured in days. If $y(t)$ is the population at time t, we know that $y' = 0.03y$ and $y(0) = 100$. Thus, $y(t) = 100e^{0.03t}$. We are asked to find the time T when $y(T) = 1000$. Thus we want $y(T) = 100e^{0.03T} = 1000$. Thus we need $e^{0.03T} = 10$ or $T = \ln(10)/0.03 \approx 76.753$ days.

To find the doubling time, we solve

$$100e^{0.03(t+T)} = 2 \cdot 100e^{0.03t}$$

for T to find $e^{0.03T} = 2$ or $T = \ln(2)/0.03$ days. ∎

1.4 Example: Carbon dating

The carbon in the atmosphere includes two isotopes C_{14} and C_{12} and the ratio of these isotopes in a living plants and animals roughly the same as in the atmosphere. When the organism dies, the C_{14} starts to decay. If $R(t)$ represents the ratio of C_{14} to C_{12} at a time t years after the organism's death, we find that $R(t) = R_a e^{-kt}$ where R_a is ratio of C_{14} to C_{12} in the atmosphere. The half-life of $C14$ is approximately 5730 years.

Example. Suppose that in a sample of wood, the ratio of $C14$ to C_{12} is 23% of the ratio in the atmosphere. How long ago was the wood in a living tree?

Solution. Let $R(t)$ the ratio of C_{14} to C_{12} at a time t years after the tree dies. As $R(t) = R_a e^{-kt}$, we want to find the time T so that $R(T) = 0.23R_a$.

Before we can do this, we need to find k. We use that the half-life is 5730 years to find k. Since $e^{-k5730} = \frac{1}{2}$, we have that $-k = \ln(1/2)/5730$ of $k = \ln(2)/5730 \approx 1.21 \cdot 10^{-4}$. Thus, if $e^{-kT} = 0.23$, we may solve for T and find that $T = \ln(0.23)/(-k) = \ln(0.23)5730/\ln(2) \approx 12,149$ years. ∎