1 Lecture 01: Functions

- Functions, domain, and range.
- Types of functions
- Graphs of lines and parabolas

The function is a central idea in mathematics. Given two sets D and Y, a *function* f takes in elements of the set D and returns a value in the set Y. We usually write $f(x)$ for the value of f at x. The set D is called the *domain* of the function. Inside the set Y is the *range* of f, the set $R = \{y \in Y : y = f(x) \text{ for some } x \in D\}$.

In this course, we will usually consider functions whose domain is the real number line $\mathbf{R}$ or a union of intervals in the real line. We may specify the domain of the function as part of the definition, in some problems physical constraints will limit the domain, otherwise the domain is the largest set of real numbers where the formula is defined. (Some books call this the natural domain of the function; ours does not provide a name for this object.)

Example. Give the domain and range of the functions

$$f(x) = 2 - \sqrt{x+3}, \qquad g(x) = 2 + \frac{1}{x+1}.$$

Give your answer as a set and using interval notation.

Solution. The square root function $\sqrt{t}$ is defined for $t \geq 0$. Thus $\sqrt{x+3}$ will be defined for $x + 3 \geq 0$ or $x \geq -3$. The domain of f is $(-3, \infty]$ or $\{x : x \geq -3\}$. To find the range, note that $\sqrt{t}$ is always positive or zero. Thus $2 - \sqrt{x+3}$ will take values which are 2 or less. The range of f is

$$(-\infty, 2] \qquad \text{or} \{x : x \leq 2\}.$$

For the function g, a problem arises when we try to divide by zero. The formula defining g makes sense if $x + 1 \neq 0$ or $x \neg -1$. Thus domain of g is

$$(-\infty, -1) \cup (-1, \infty) \qquad \text{or} \qquad \{x : x \neq -1\}.$$

To find the range, we can ask if we can solve the equation $y = 2 + 1/(x+1)$. Solving we find

$$\begin{aligned}
y &= 2 + \frac{1}{x+1} && \text{subtract 2 from each side} \\
y - 2 &= \frac{1}{x+1} && \text{multiply by } x+1 \text{ and divide by } y-2 \\
x + 1 &= \frac{1}{y-2} \\
x &= -1 + \frac{1}{y-2}
\end{aligned}$$

Thus, we see that we can find a value x with $g(x) = y$ when the above steps are right. The one place a problem might arise is when we divide by $y - 2$. We cannot divide by zero so we must have $y \neq 2$. Thus the range is

$$(-\infty, 2) \cup (2, \infty) \qquad \text{or} \qquad \{y : y \neq 2\}.$$

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1.1 Types of functions

Throughout the semester, we will study several different classes of functions. We give examples of each.

A *polynomial* function a sum of terms of the form $a_k x^k$ where k is a whole number, $0, 1, 2, \ldots$. Examples include

$$1, \qquad 2x + 3, \qquad 4x^3 - 5x + 6.$$

The natural domain of a polynomial is always the real line $\mathbf{R} = (-\infty, \infty)$. A *rational function* is a quotient of polynomials. Examples include

$$x + 1, \qquad \frac{7x^2 + 8}{x^3 - 27}, \qquad \frac{2x + 1}{x^2 + 1}.$$

The first function is a rational function since it can be thought of as the quotient of $x + 1$ and 1. The domain of the rational function is the set where the denominator (or bottom) of the fraction does not vanish.

The class of algebraic functions is complicated to define. Examples include polynomials and rational functions. In addition, the square root, cube root, higher roots are algebraic. Examples include

$$\sqrt{x - 1}, \qquad \sqrt[3]{x^2 - x}, \qquad \sqrt{\frac{x + 1}{x - 1}}.$$

To find the domain of such a function, remember that we cannot take an even root of a negative number. Thus the domain of $\sqrt{x - 1}$ is $[1, \infty)$ while the domain of $\sqrt[3]{x^2 - x}$ is all real numbers.

The most interesting classes of functions are the transcendental functions. These include the trigonometric functions, $\sin(x)$, $\cos(x)$, $\tan(x)$, and $\sec(x)$. The exponential and logarithm functions are also transcendental functions. We will talk about these more in later lectures.

1.2 Graphs

Given an equation in two variables, usually x and y, the *graph* is the set of points which satisfy the equation. Turning functions into graphs by considering the equation

$y = f(x)$ will allow us to connect ideas in calculus with geometric properties of the graph. Today we will consider two important classes of graphs, lines and parabolas.

A *line* is given by an equation of the form $Ax + By = C$ with A and B not both zero. Horizontal lines are of the form $y = C$ and vertical lines are of the form $x = C$. To talk about the remaining lines, we need the idea of *slope*. Given two points in the plane (x_1, y_1) and (x_2, y_2) the slope of the line segment joining them is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

A line has the property that any two points on the line have the same slope. Thus we can determine a line by specifying a slope and any point on the line.

Point slope form of a line. The collection of all points that form a slope m with (x_1, y_1) is given by

$$m = \frac{y - y_1}{x - x_1}.$$

We can rewrite this as

$$y - y_1 = m(x - x_1).$$

An important special case is when the point is the y-intercept $(0, b)$ where the line crosses the y axis. This gives us the *point intercept* form of the line

$$y = mx + b.$$

Example. Find the line which passes through $(-5, 3)$ and is perpendicular to $x+2y = 4$.

Solution. We will need to recall the fact that two lines are perpendicular if the product of their slopes is -1. To find the slope of the line $x + 2y = 4$, we rewrite it in point intercept form to obtain

$$y = -\frac{1}{2}x + 2$$

and see that the slope is $-1/2$. Taking the negative reciprocal, the slope of the perpendicular line is 2. Using the point slope form, the line with slope 2 that passes through $(-5, 3)$ has the equation

$$y - 3 = 2(x + 5)$$

or simplifying gives

$$y = 2x + 13.$$

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Finally, we discuss parabolas briefly. A parabola is the graph of an equation of the form $y = ax^2 + bx + c$ with $a \neq 0$. Using the procedure of completing the square we may put the parabola in the form

$$y - k = a(x - h)^2$$

where the point (h, k) is the extreme point on the parabola. This is the point with smallest y-coordinate if $a > 0$ and largest y-coordinate if $a < 0$.

We give an example that leads to a parabola.

Example. Find the rectangle with perimeter 100 meters and largest possible area.

Solution. If ℓ and w are the lengths of the two sides, we may write the area as

$$A = \ell w.$$

Since we know that the perimeter is 100 we have $2\ell + 2w = 100$ or solving for w, $\ell = 50 - w$. Substituting we find that

$$A = 50w - w^2.$$

Completing the square, we write

$$A = -(w^2 - 50w + 25^2 - 25^2) = -(w - 25)^2 + 625.$$

We chose to add and subtract 25^2 since 25 is half of the coefficient of w and then the first three terms inside the parentheses give the square $(w-25)^2$. Since the first term is negative or zero, the largest area is $A = 625$ square meters. ∎

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