1 Lecture 03: Inverse functions

- One to one functions and inverse functions
- The exponential and logarithm functions
- The number e and the natural logarithm
- Inverse trig functions

1.1 Definitions

We say that a function is *one to one* on a domain D if for every value c, the equation $f(x) = c$ has at most one solution.

Example. Which of the functions $f(x) = x^2$ and $g(x) = 1/x$ are one-to-one?

Solution. The equation $x^2 = c$ has two solutions when $c > 0$, namely $+\sqrt{c}$ and $-\sqrt{c}$. Thus the function $f(x) = x^2$ is not one-to-one on the real line.

But the equation $1/x = c$ has only the solution $x = 1/c$, unless $c = 0$, and then it has no solutions. Thus g is one-to-one on the domain $(-\infty, 0) \cup (0, \infty)$.

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Definition. If f is a function with domain D and range R and g is a function with domain R and range D so that $f \circ g(x) = x$ for all x in R and $g \circ f(x) = x$ for all x in D, then we say that the function f is *invertible* and g is the inverse of g. We use the notation f^{-1} for the inverse of f.

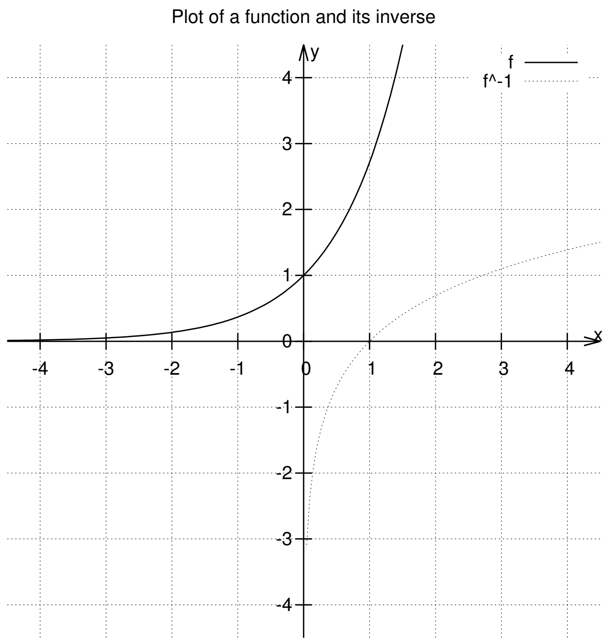
Example. Let $f(x) = 1 + x^2$ with domain $(-\infty, 0]$. Show that f has an inverse and give the domain and range of f^{-1}.

Solution. We first note that the domain of f is $(-\infty, 0]$ and the range is $[1, \infty)$. If $f(x) = y$, then we will want $f^{-1}(y) = x$. Thus we need to solve the equation $y = 1 + x^2$ for $y \geq 1$. The solutions are $x = \pm\sqrt{y-1}$. Since the range of the inverse function is to be $(-\infty, 0]$, we choose the negative sign. Thus

$$f^{-1}(x) = -\sqrt{y-1}.$$

The domain of f^{-1} is $[1, \infty)$, the range of f and the range of f^{-1} is $(-\infty, 0]$, the domain of f. ∎

If we have $f(x) = y$, then the inverse function must take y back to x and if this is to determine $f^{-1}(y)$ uniquely, we must have that f is one-to-one. Furthermore, we can only hope to define $f^{-1}(y)$ for values y in the range of f. This observation also helps us to graph the inverse function. If (x, y) is a point on the graph of f, then (y, x) gives a point of the graph of f^{-1}. The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.



1.2 Exponential and logarithms

An important example of a function and its inverse are the exponential and logarithm functions. If $a > 0$ and $a \neq 1$, we may define the *exponential function with base* a by $f(x) = a^x$. We are familiar with the powers $a^n = a \cdots a$ as repeated multiplication if $n = 1, 2, \ldots$ and $a^{-n} = 1/a^n$ if $n = 1, 2, \ldots$. Since $(a^{1/n})^n = a$, we want $a^{1/n}$ to be the nth root $\sqrt[n]{a}$ and finally we can put $a^{m/n} = (a^{1/n})^n$. However the definition of a^x for x an irrational number is more subtle. By the end of this course, we will be able to say more about this. Altogether, we have the exponential function $f(x) = a^x$ is defined for x in the domain $(-\infty, \infty)$ and has range $(0, \infty)$.

The exponential function satisfies the properties

$$a^0 = 1 \tag{1}$$
$$a^1 = a \tag{2}$$
$$a^x a^y = a^{x+y} \tag{3}$$
$$a^{-x} = 1/a^x \tag{4}$$
$$(a^x)^y = a^{xy} \tag{5}$$

$$ \tag{6} $$

The inverse function to this exponential function is called the logarithm with base a, denoted by $\log_a$.

Example. Find the values of $\log_{10}(100)$ and $\log_2(\sqrt{2})$.

Solution. Since the function 10^x takes 2 to 100, the inverse function takes 100 to 2, $\log_{10}(100) = 2$.
Since $\sqrt{2} = 2^{1/2}$, we have that $\log_2(\sqrt{2}) = 1/2$. ∎

Each of the properties of the exponential function can be recast as a property of the logarithm function.

$$
\begin{aligned}
\log_a(1) &= 0 && (7)\\
\log_a(a) &= 1 && (8)\\
\log_a(x) + \log_a(y) &= \log_a(xy), \qquad x > 0, y > 0 && (9)\\
\log_a(1/x) &= -\log_a(x), \qquad x > 0 && (10)\\
\log_a(x^r) &= r\log_a(x), \qquad x > 0, r \in (-\infty, \infty). && (11)
\end{aligned}
$$

To see why (9) is true, we can write $x = a^{\log_a(x)}$, $y = a^{\log_a(y)}$, and $xy = a^{\log_a(xy)}$. Then using property (3) of the exponential function, we have

$$a^{\log_a(xy)} = xy = a^{\log_a(x)}a^{\log_a(y)} = a^{\log_a(x)+\log_a(y)}.$$

Since the exponential function is one-to-one, we have $\log_a(xy) = \log_a(x) + \log_a(y)$.

1.3 The number e

As we saw in the previous section, there is a different logarithm $\log_a$ for each $a > 0$ (except $a = 1$). Which one is best? For the purposes of calculus, we use a special number $e \approx 2.781828...$ The logarithm and exponential function for this base are written as

$$e^x \text{ or } \exp(x) \qquad \text{and} \qquad \ln(x)$$

and $\ln(x)$ is called the natural logarithm. We will see why this logarithm is natural. At the moment, it is a puzzle why we use the base e instead of a more familiar number such as 2 or 10. However, we will see that

It is useful to observe that any exponential function can be expressed in terms of the function e^x.

Example. Write 4^x in the form e^{rx}.

Solution. We want to have $4^x = e^{rx}$. If we take the natural log of both sides, we have $x\ln(4) = rx$ or $r = \ln(4)$. Thus we have $4^x = e^{x\ln(4)}$. ∎

Example. If x and y are positive numbers and $\ln(xy^2) = 2$ and $\ln(x/y) = 0$, find x and y.

Solution. If we write $x = e^a$ and $y = e^b$, (actually $a = \ln(x)$ and $b = \ln(y)$ then we have

$$2 = \ln(e^a e^{2b}) = (a + 2b)\ln(e) = (a + 2b)$$

and

$$0 = \ln(e^a/e^b) = \ln(e^{a-b}) = a - b.$$

Solving the system of equations

$$a + 2b = 2, \qquad a - b = 0$$

gives $a = 2/3$ and $b = 2/3$ or $x = y = e^(2/3)$.

And of course we can check our answers. ∎

1.4 Inverse trigonometric functions

None of the trigonometric functions are one-to-one. However, by restricting the domain we obtain a one-to-one function. The standard choices for domains are below:

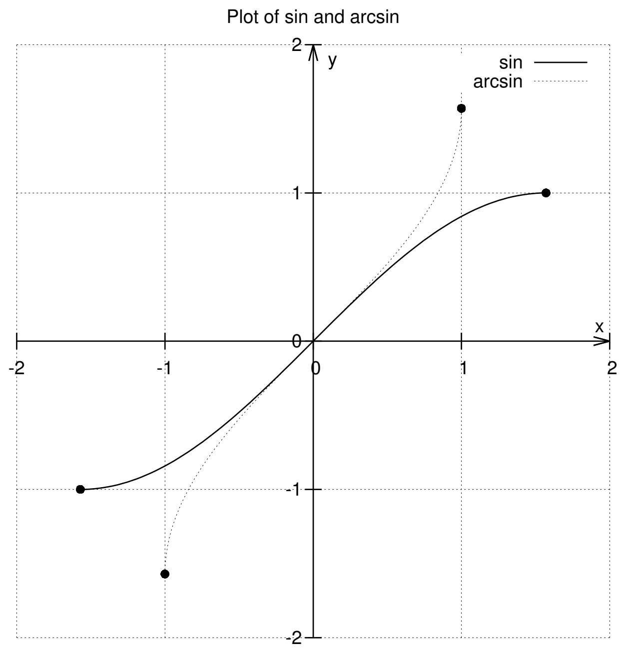
$\sin(x)$	$[-\pi/2, \pi/2]$
$\cos(x)$	$[0, \pi]$
$\tan(x)$	$(-\pi/2, \pi/2)$
$\sec(x)$	$[0, \pi/2) \cup (\pi/2, \pi]$

The inverse function to sin on the domain $[-\pi/2, \pi/2]$ will be denoted using either the notation $\sin^{-1}$ or arcsin. The prefix arc suggests that when we find $\theta = \arcsin(x)$, we are looking for the angle or arc which has $\sin(\theta) = x$. Similar considerations apply to arccos or $\cos^{-1}$, arctan or $\tan^{-1}$, and arcsec or $\sec^{-1}$.

Note that there is an inconsistency in our use of the notation $\sin^{-1}$. The sin function does not have an inverse. Rather, we are taking the inverse of the function g with $g(x) = \sin(x)$ for x in the domain $[-\pi/2, \pi/2]$. The notation $\sin^{-1}$ is ambiguous because it is not clear if it represents the inverse function arcsin or the reciprocal $1/sin = \csc$. Perhaps the next time we invent mathematics, we can find better notation.

Example. Sketch the graph of $\arcsin(x)$. Give the domain and range. Find $\arcsin(1/2)$.

Solution. We sketch the graph of $y = \sin(x)$ for x in $[-\pi/2, \pi/2]$. Several convenient points on the graph include $(-\pi/2, -1)$, $(0, 0)$, and $(\pi/2, 1)$. Thus the points $(-1, -\pi/2)$, $(0, 0)$, and $(1, \pi/2)$ lie on the graph of $\sin^{-1}$ or arcsin. Plotting these points and doing our best to fill in the intermediate points gives the following plot.



We are considering $\sin(x)$ with domain $[-\pi/2, \pi/2]$ and range $[-1, 1]$. The inverse function arcsin will have domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$.

To find $\arcsin(1/2)$, we recall that in a $\pi/6$, $\pi/3$, $\pi/2$ triangle, with hypotenuse 1, the legs are of length $1/2$, $\sqrt{3}/2$ and $\sin(\pi/6) = 1/2$. Thus $\arcsin(1/2) = \pi/6$.

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