1 Lecture 06: Continuous functions

- Definitions: Continuity, left continuity, right continuity, continuity on intervals
- Examples-jump discontinuities, infinite discontinuities, other
- Continuous functions and limits

1.1 Definitions

Definition. Let f be a function defined on an open interval containing a. We say that f *is continuous at* a if we have

$$\lim_{x\to a} f(x) = f(a).$$

Example. Let

$$f(x) = \begin{cases} kx, & x > 2 \\ x^2, & x \le 2. \end{cases}$$

Find k so that f is continuous at 2.

Solution. For f to be continuous at 2, we need that $\lim_{x\to 2} f(x)$ to exist and equal $f(2) = 4$. Since the function is defined piecewise, it is natural to look at the left and right limits separately. From our limit laws, we have

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 = 4 \qquad \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} kx = 2k.$$

Thus for the limit to exist, we need $2k = 4$ or $k = 2$ and then $\lim_{x\to 2} f(x) = 4$. Since $f(2) = 4$, f will be continuous for this choice of k. ∎

We also need the following variants of continuity.

- A function f is *left-continuous at* c if $\lim_{x\to c^-} f(x) = f(c)$.
- A function f is *right-continuous at* c if $\lim_{x\to c^+} f(x) = f(c)$.
- A function f is *continuous on an open interval* (a, b) if f is continuous at each point c in the interval.
- A function f is *continuous on a closed interval* $[a, b]$ if f is continuous at each point c in the interval (a, b), right-continuous at a, and left-continuous at b.

1.2 Some discontinous functions

Example. Let

$$f(x) = \begin{cases} x/|x|, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

What can you say about the continuity of f.

Solution. It is easy to see that if $a > 0$ $\lim_{x \to a} f(x) = 1 = f(a)$ and if $a < 0$, then $\lim_{x \to a} f(x) = -1 = f(a)$.

At $x = 0$, we have

$$\lim_{x \to 0^+} f(x) = 1, \qquad \lim_{x \to 0^-} f(x) = -1.$$

Thus the limit does not exist and the function cannot be continuous. ∎

When a function is as in the previous example: the one-sided limits at a exist and are different, we say that f has a *jump discontinuity* at a.

A second example is that the one-sided limits do not exist, but are $+\infty$ or $-\infty$. Then we say that the function has an *infinite discontinuity.* An example is the function

$$f(x) = \frac{1}{x-8}$$

which has an infinite discontinuity at 8.

Exercise. Let

$$f(x) = \begin{cases} \sin(\pi/x), & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Determine if f is continuous at 0.

1.3 Continuous functions

Most of the functions we work with every day in calculus are continuous on their domains.

We give a list

- Polynomials are continuous on $(-\infty, \infty)$
- Exponential functions a^x are continuous on $(-\infty, \infty)$.
- $\sin(x)$ and $\cos(x)$ are continuous on the real line $(-\infty, \infty)$

The following rules for combining continuous functions give us many more continuous functions.

- If f and g are continuous at a, then $f + g$ and fg are continuous at a. If, in addition, $g(a) \neq 0$, then f/g is continuous at a.
- If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.
- If f is continuous on an open interval I with domain R and f^{-1} exists, then f^{-1} is continuous on R.

Using these rules, we obtain a number of other continuous functions.

- Rational functions are continuous on their domain. This follows since a rational function is a quotient of polynomials.
- A logarithm function $\log_a(x)$ is continuous on $(0, \infty)$. This follows since the logarithm $\log_a(x)$ is the inverse of a^x.
- If $n = 3, 5, \ldots$ is an odd positive integer $\sqrt[n]{x}$ is continuous on $(-\infty, \infty)$ This follows since $\sqrt[n]{x}$ is the inverse of x^n.
- If $n = 2, 4, \ldots$ is an even positive integer $\sqrt[n]{x}$ is continuous on $[0, \infty)$. This follows since $\sqrt[n]{x}$ is the inverse of x^n on the domain $[0, \infty)$, for n even.
- $\tan(x)$ is continuous $\{x : x \neq (2k+1)\pi/2 \text{ for } k \neq 0, \pm 1, \pm 2, \ldots\}$. This follows since $\tan(x)$ is the quotient of $\sin(x)$ and $\cos(x)$.

Exercise. Determine where $\cot(x)$, $\csc(x)$, and $\sec(x)$ are continuous.

Once we know a function f is continuous, we may use the definition of continuity,

$$\lim_{x \to a} f(x) = f(a)$$

to evaluate a limit. This is known as the *direct substitution rule.* Be sure to note that a function is continuous before applying the direct substitution rule.

Example. Use the direct substitution rule to evaluate the limits

$$\lim_{x \to 0} \frac{e^x + 1}{e^{2x} + 1} \qquad \lim_{x \to 0} \frac{e^x - 1}{e^{2x} - 1}.$$

Solution. Since $(e^x + 1)/(e^{2x} + 1)$ is continuous at $x = 0$, we may use the direct substitution rule to find $\lim_{x \to 0} \frac{e^x+1}{e^{2x}+1} = 1$. For the second limit, we notice that $g(x) = \frac{e^x-1}{e^{2x}-1}$ is undefined at $x = 0$. However, if we factor, we may write

$$\frac{e^x - 1}{e^{2x} - 1} = \frac{e^x - 1}{(e^x - 1)(e^x + 1)} = \frac{1}{e^x + 1}.$$

The function $1/(e^x + 1)$ is continuous at $x = 0$. Thus we have

$$\lim_{x \to 0} \frac{e^x - 1}{e^{2x} - 1} = \lim_{x \to 0} \frac{1}{e^x + 1} = 1/2.$$

∎