

1 Lecture 20: Implicit differentiation

1.1 Outline

- The technique of implicit differentiation
- Tangent lines to a circle
- Examples

1.2 Implicit differentiation

Suppose we have two quantities or variables x and y that are related by an equation such as

$$x^2 + 2xy^2 + x^3y = xy.$$

If we know that $y = y(x)$ is a differentiable function of x , then we can differentiate this equation using our rules and solve the result to find y' or dy/dx . In this course, we will not learn conditions which guarantee that y is a differentiable function of x . This is a topic for a later course. This assumption is usually valid and the technique is very useful.

We begin with a simple example where we use this technique to find tangent lines to a circle.

Example. Consider the circle centered at the origin with radius 5 which is the set of points (x, y) which satisfy $x^2 + y^2 = 25$.

Find dy/dx on the circle.

Find the tangent lines at the points on the circle with x -coordinate 4.

Show that a tangent line to the circle is perpendicular to the radius at the point of tangency.

Solution. We imagine that $y = y(x)$ is a function of x in the equation defining the circle and differentiate both sides with respect to x .

$$\begin{aligned}\frac{d}{dx}(x^2 + y(x)^2) &= \frac{d}{dx}25 \\ 2x + 2y\frac{dy}{dx} &= 0.\end{aligned}$$

Observe that when we differentiated the term $y(x)^2$, we used the chain rule with $y(x)$ as the inside function. Next, we solve this equation for dy/dx to find

$$\frac{dy}{dx} = -x/y.$$

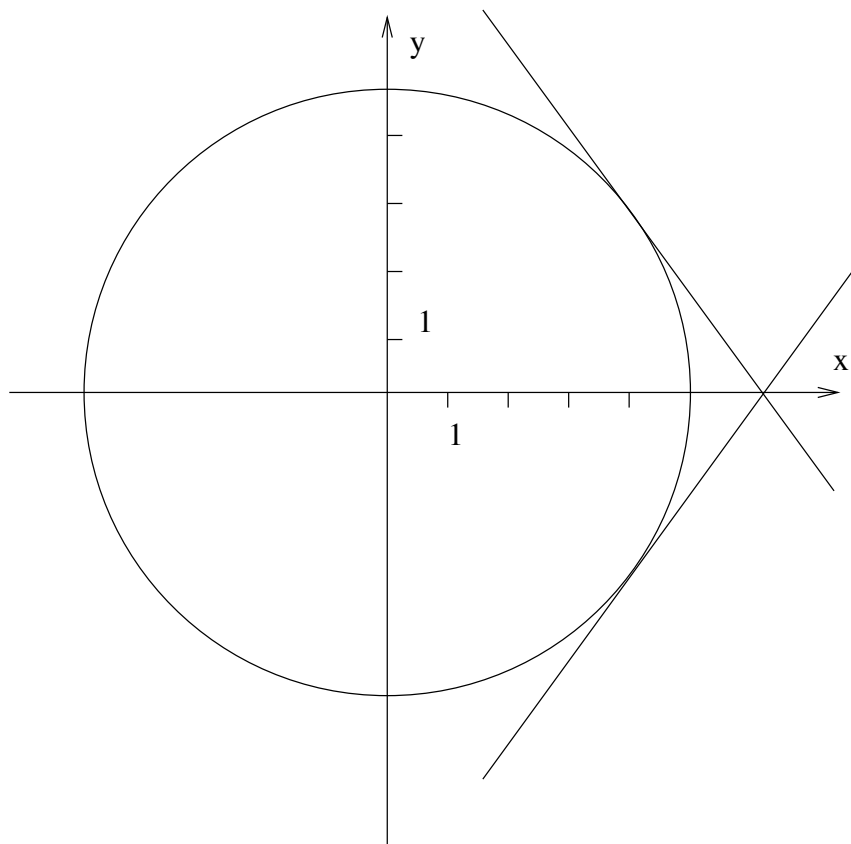
To find the tangent lines when the x -coordinate is 4, we solve $4^2 + y^2 = 25$ for y to find that $y = 3$ or -3 . Thus there are two tangent lines we need to find. One

passes through the point $(x, y) = (4, 3)$ and has slope $dy/dx = -4/3$. The second passes through $(x, y) = (4, -3)$ and has slope $dy/dx = 4/3$. The point slope forms of the equation are:

$$y - 3 = -\frac{4}{3}(x - 4)$$

$$y + 3 = \frac{4}{3}(x - 4)$$

The following sketch shows the tangent lines and the circle and helps to check our answer.



Next, at a general point (x, y) on the circle the tangent line has slope $-x/y$ while the radius which is the line segment joining (x, y) to $(0, 0)$ has slope y/x . The product of these slopes is -1 and hence the lines are perpendicular.

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Exercise. We can also find tangent lines by solving the equation $x^2 + y^2 = 25$ to give $y = \pm\sqrt{25 - x^2}$ and then using techniques we learned earlier.

Carry this out to check your answer to the previous problem.

Example. Find the second derivative y'' at the point $(3, 4)$ on the circle $x^2 + y^2 = 25$.

Solution. We begin as before by differentiating $x^2 + y^2 = 25$ with respect to x and obtain

$$2x + 2yy' = 0. \quad (1)$$

As before, we have $y' = -x/y$. Want to differentiate again. It is probably simpler to differentiate (1) rather than $y' = -x/y$ to avoid using the quotient rule. Differentiating both sides of (1) with respect to x and using the product rule on the second term gives

$$2 + 2yy'' + 2(y')^2 = 0.$$

Solving for y'' gives

$$y'' = -(x + (y')^2)/y = -\frac{1}{y} - \frac{x^2}{y^3}.$$

In the second step we used that $y' = -x/y$. Now we may substitute the values $(x, y) = (3, 4)$ to obtain

$$y'' = -1/4 - 9/64 = -25/64.$$

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Example. Find the tangent line to the curve defined by $x^2 + 2y^2 = 2 + x^2y$ at the point $(x, y) = (3, 1)$.

Solution. The tangent line will go through the given point $(3, 1)$ thus the only thing we need to find is the slope, y' . We visualize that $y = y(x)$ is a function of x and differentiate both sides of the equation

$$(x^2 + 2y(x)^2)' = (2 + x^2y(x))'$$

where $'$ denotes the derivative with respect to x . We use the product and chain rules to conclude

$$2x + 4yy' = 0 + 2xy + x^2y'.$$

We solve this equation for y' and obtain

$$y'(4y - x^2) = 2xy - 2x \text{ or } y' = \frac{2xy - 2x}{4y - x^2}.$$

Substituting $(x, y) = (3, 1)$ gives

$$y' = \frac{6 - 6}{4 - 9} = 0.$$

Thus the tangent line is the line through $(3, 1)$ with slope 0 which gives

$$y = 1.$$

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In our last example, we will not use x and y . It is useful to remember that the technique of implicit differentiation can be used to find the rate of change between any two variables.

Example. Consider the quadratic equation

$$x^2 + 2x + c = 0.$$

- a) Find the roots when $c = 0$.
- b) Find the derivative of x with respect to c and for each root from part a) determine if the root increases or decreases as c increases.
- c) Sketch the parabola $y = x^2 + x + c$ for $c = 0$ and check if your answer in part b) makes sense.

Solution. a) When $c = 0$, the equation $x^2 + 2x = 0$ factors as $x(x + 2) = 0$. The roots are $x = 0$ and $x = -2$.

b) We differentiate the equation with respect to c and find

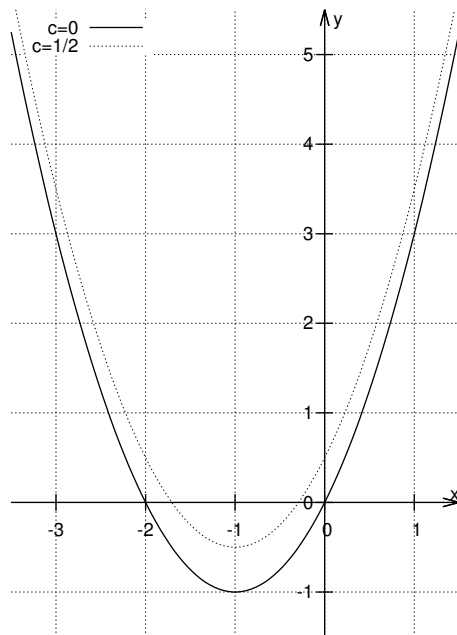
$$2x \frac{dx}{dc} + 2 \frac{dx}{dc} + 1 = 0.$$

Solving for the derivative gives

$$\frac{dx}{dc} = -\frac{1}{2x + 2}.$$

At $x = 0$, we have $dx/dc = -1/2$ so this root decreases as c increases. At $x = -2$, we have $dx/dc = 1/2$ so this root increases.

c) As c increases, the parabola is shifted up and the roots move towards $x = -1$.



Exercise. Let $y = \tan^{-1}(x)$ be the inverse tangent or arctangent function. Find the derivative dy/dx by applying implicit differentiate to the equation

$$x = \tan(y).$$

This provides another way to understand our method for finding derivatives of inverse functions.

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