1 Lecture 27: The shape of a graph

1.1 Outline

- Concavity and inflection points
- Second derivative test for local extrema

1.2 Concavity and inflection points

Definition. We say that a function f is concave up on an open interval (a, b) if f' is increasing on the interval.

We say that a function f is concave down on an interval if f' is decreasing on the interval.

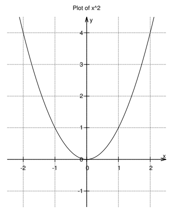
Definition. A point $(c, f(c))$ on the graph of f is a point of inflection if f changes concavity at c.

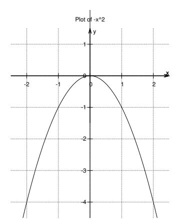
Example. Consider the functions $f(x) = x^2$, $g(x) = -x^2$ and $h(x) = x^3$. Determine the intervals of concavity and find inflection points.

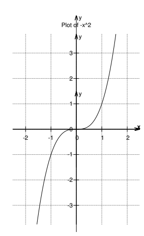
Solution. Since $f'(x) = 2x$, f' is increasing everywhere and f is concave up.

Since $g'(x) = -2x$, f' is decreasing everywhere and f is concave down.

Since $h'(x) = 3x^2$, h' is decreasing on $(-\infty, 0)$ and thus concave down there. Also, h' is increasing on $(0, \infty)$ and concave up there. Since h changes concavity at 0, $(0, h(0))$ is an inflection point.







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We should keep these three examples in mind to help illustrate the definitions.

An immediate consequence of our first derivative test for concavity is the following second derivative test for concavity.

Theorem 1 *Suppose that f is a function and the second derivative f'' exists on the interval (a, b).*

- *If $f''(x) > 0$ for all x in the interval, then f is concave up.*
- *If $f''(x) < 0$ for all x in the interval, then f is concave down.*
- *If c is in the interval (a, b), $f''(c) = 0$ and $f''(x)$ changes sign at $x = c$, then f has an inflection point at $(c, f(c))$.*

Proof. If $f''(x) > 0$ for all x in an open interval, then f' is increasing in the interval. Thus f is concave up.

The proof for functions with f'' negative is the same.

The characterization of inflection points follows from the test for concavity and the definition of an inflection point. ∎

Example. Let $f'(x) = x^3 - 3x^2$. Determine the intervals on which f is increasing or decreasing.

Find the critical points and determine if f has a local maximum or minimum at each of them. Determine the intervals on which f is concave up and concave down and find any inflection points.

Solution. Note well that we are given f', not f. The function $f'(x) = x^2(x-3) = 0$ if $x = 0$ and $x = 3$. Testing signs, we see that

	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
x^2	$+$	$+$	$+$
$(x-3)$	$-$	$-$	$+$
$x^2(x-3)$	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

The function is increasing on $(-\infty, 0)$ and $(0, 3)$ from the first derivative test. It is also increasing on the interval $(-\infty, 3)$, but this does not follow from our first derivative test. The function is decreasing on the interval $(3, \infty)$.

From the first derivative test for local extremes, we see that f has a local minimum at $x = 3$. The critical point at $x = 0$ is not a local extremum.

To compute the second derivative, we first multiply out $f'(x) = x^3 - 3x^2$. We have $f''(x) = 3x^2 - 6x = 3x(x-2)$, we obtain that $f''(x) = 0$ at $x = 0$ and $x = 2$. Checking the signs of f'' we have

	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
$f''(x) = 3x(x-2)$	$+$	$-$	$+$
Behavior of f'	increasing	decreasing	increasing
Behavior f	concave up	concave down	concave up

From this table we can see that f has two inflection points or points where it changes concavity. At $(0, f(0))$, f changes from concave up to down and at $(2, f(2))$, f changes from concave down to up. ∎

1.3 Second derivative test for local extrema

Finally, the second derivative gives us a very simple test for local extrema.

Theorem 2 *Suppose that f is a function defined in an open interval containing c and $f'(c) = 0$*

- *If $f''(c) > 0$, then c is a local minimum for f.*
- *If $f''(c) < 0$, then c is a local maximum for f.*

We will omit the proof.

Example. Let $f(x) = x^4$. Which test can we use to determine that 0 is a local minimum. Does f have an inflection point at 0?

Solution. We compute $f''(x) = 12x^2$. The second derivative test fails as $f''(0) = 0$. The first derivative test is still useful as we have $f'(0) = 0$, $f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$. Thus the first derivative tells us that f has a local minimum at 0.

Since $f''(x) > 0$ if $x > 0$ or if $x < 0$, the function f is concave up on $(-\infty, 0)$ and $(0, \infty)$. Since there is no change in concavity, 0 is not an inflection point. It is also correct to say that f is concave up on $(-\infty, \infty)$, but this does not follow from our second derivative test. ∎

Exercise. What are the intervals of concavity for $f(x) = -1/x$?

Exercise. Give the intervals of monotonicity and concavity for the function

$$f(x) = x/(1 + x^2).$$

Find local maxima and minima and inflection points.

Solution. The derivative is

$$f'(x) = \frac{1 - x^2}{(1 + x^2)^2}.$$

We have the following sign table

	$(\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$f'(x)$	$-$	$+$	$-$
f	decreasing	increasing	decreasing

From this we can see that there will be a local minimum at $x = -1$ and a local maximum at $x = 1$. We have $f(-1) = -1/2$ and $f(1) = 1/2$.

For the second derivative,

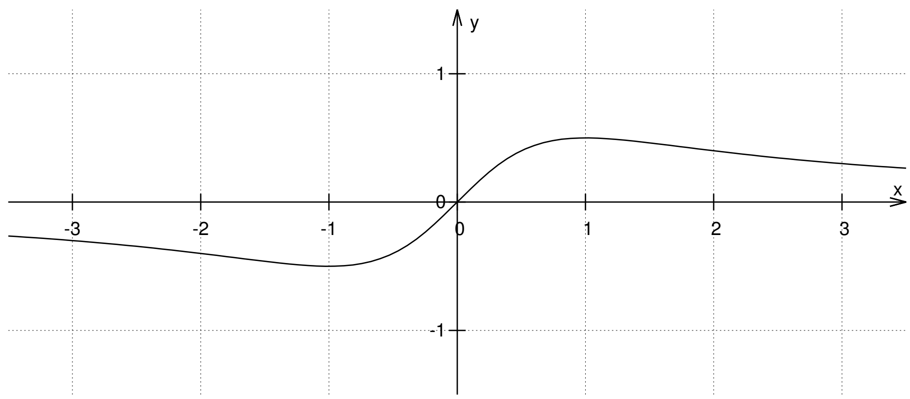
$$f''(x) = \frac{2x(x^2 - 3)}{(1 + x^2)^3}.$$

The following table describes the concavity,

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
f''	$-$	$+$	$-$	$+$
f	concave down	concave up	concave down	concave up

We have inflection points at $(-\sqrt{3}, -\sqrt{3}/4)$, $(0, 0)$, and $(\sqrt{3}, \sqrt{3}/4)$.

A look at the graph confirms this information,



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Exercise. In a problem asking for the intervals where $f(x) = x^3 - 3x$ is increasing, Webwork expects the answer to be $(-\infty, -1) \cup (1, \infty)$.

Why is this not right?

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