1 Lecture 31: Newton's method

- Explanation of the method.
- Examples, applications
- What could go wrong?

1.1 Newton's method

Newton's method is a procedure for approximating the solutions of an equation $f(x) = 0$. These are also called the zeros of f. The method is most useful when simpler algebraic methods such as the quadratic formula do not apply.

Newton's method is an iterative procedure that finds an approximation to a root of $f(x) = 0$ and then uses the approximation to help find a better approximation. This gives a list of numbers $x_1, x_2, x_3, \ldots$ which we call successive approximations. We begin by explaining the main step of method with shows us how to find x_{n+1} from x_n.

Suppose that x_n is a guess for a solution of $f(x_n) = 0$. We construct the linearization of f at x_n and then solve $L(x) = 0$ to obtain the next approximation. Since the linearization is

$$L(x) = f(x_n) + f'(x_n)(x - x_n),$$

the solution of $L(x) = 0$ is

$$x = x_n - \frac{f(x_n)}{f'(x_n)}.$$

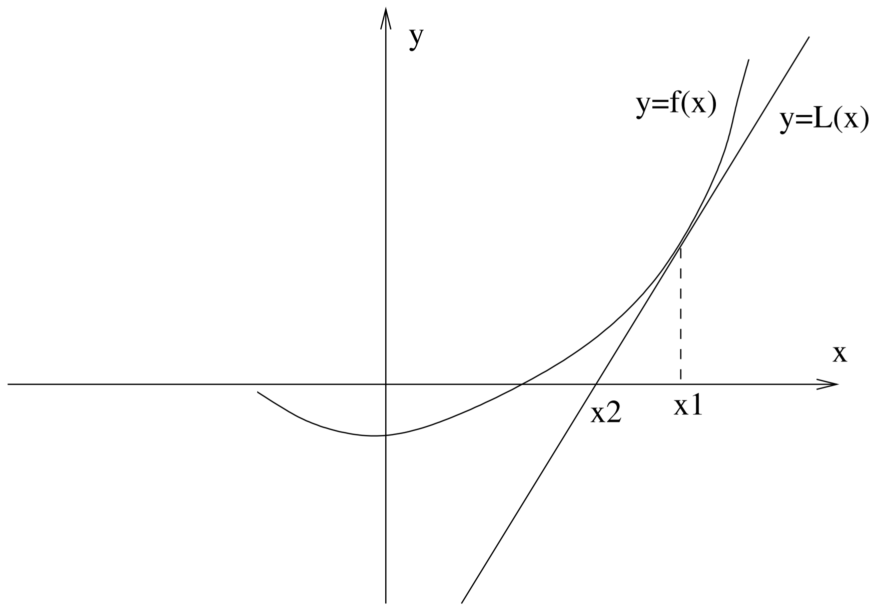
Thus Newton's method is the following:

- Guess a starting point x_1.
- Given an approximation x_n, compute the next approximation by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

- Continue until tired.

The picture gives one step of Newton's method.



In some examples, the starting point will be important. For example, different starting points will lead to different zeros, or some starting points will lead to a sequence of successive approximations that do not approach a number. If we have some information about the solutions, then we should try to choose a starting point that is near the solution we want.

This method is called an iterative method as it applies a simple step repeatedly to hopefully obtain a solution.

Example. Use Newton's method to approximate $\sqrt{3}$.

Solution. We need to find a simple function with $f(\sqrt{3}) = 0$. We let $f(x) = x^2 - 3$ and observe that $f(\sqrt{3}) = 0$. To find the iteration formula, we compute $f'(x) = 2x$ and then we have

$$x_{n+1} = x_n - \frac{x_n^2 - 3}{2x_n} = x_n - \frac{1}{2}x_n + \frac{1}{2}\frac{3}{x_n} = \frac{1}{2}(x_n + \frac{3}{x_n}).$$

We may use any positive number as a starting point, $x_1 = 3$. Computing several values we find (decimal values are approximate),

n	x_n	$\sqrt{3} - x_n$
1	3	1.3
2	2	0.27
3	$7/4 = 1.75$	0.017
4	$97/56 \approx 1.7321$	9.2×10^{-5}
5	1.73205	2.4×10^{-9}
6	1.73205080756888	$< 10^{-13}$.

A couple of comments. 1) This method of computing square roots is intuitive appealing. If x_n is smaller than $\sqrt{3}$, $3/x_n$ will be larger and the average is a reasonable candidate. Similarly if x_n is too large, then $3/x_n$ will be too small. 2) What is not obvious is why this method works so well. Notice that after line 2 of the table, the error is roughly the square of the previous error. We will leave it to a later course (or the written assignment) to explain this.

Note that we were lucky with our choice of f. The computations to find $\sqrt{3}$ only involved arithmetic. Thus, this application of Newton's method lets us do something new, find square roots, using only arithmetic. ∎

To carry out Newton's method quickly a calculator or computer is useful. Here is one way to execute Newton's method on a TI calculator.

1. Enter the initial guess, x_1.
2. Enter the iteration formula `ANS` $- f($`ANS`$)/f'($`ANS`$)$ with x replaced by `ANS`. The variable `ANS` represents the answer from the last calculation and is obtained by pressing `2ND` and $(-)$.
3. Press `ENTER` repeatedly. Each press of the `ENTER` key gives the next approximation.

Example. Use Newton's method to find the smallest positive critical point of $f(x) = x^5 - 6x^2 - x$.

Solution. We want to solve $f'(x) = 0$, so we compute $f'(x) = 5x^4 - 12x - 1$ and $f''(x) = 20x^3 - 12$.

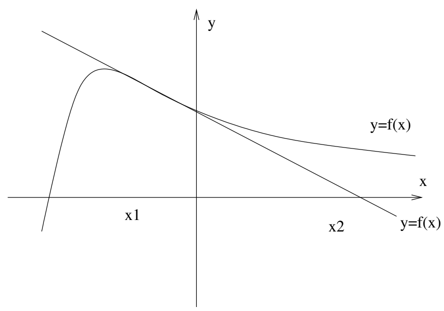
A look at the graph of f suggests that $x = 2$ is a good starting point. Thus we have that $x_{n+1} = x_n - f'(x_n)/f''(x_n)$. Applying the iteration, gives the approximations:

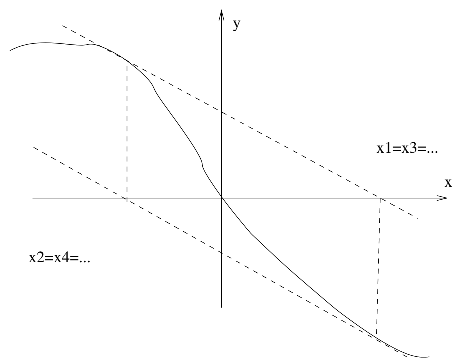
n	x_n
1	2
2	1.63
3	1.432
4	1.3712
5	1.3650931378220
6	1.36556463898463
7	1.36556463611639
8	1.36556463611639

Substituting, we can check that $f'(1.36556463611639)$ is very close to zero. ∎

1.2 What could go wrong?

Newton's method works very well when we are close to a root. However, it does not always succeed in finding a root. The graphs below suggest a couple of problems that might occur.





In the first, the successive approximations $x_1, x_2, \ldots$ run off to infinity. In the second example, the successive approximations jump back and forth between two values and never approach the zero of f.

The following result tells us that if x_n approaches a value r, then $f(r) = 0$.

Theorem 1 *That f is a differentiable function in an open interval containing r and f' is continuous with $f'(r) = 0$. If $\lim_{n\to\infty} x_n = r$, then $f'(r) = 0$.*

Proof. If we take the limit of both sides of the equation $x_{n+1} = x_n - f(x_n)/f'(x_n)$, we obtain

$$r = r - f(r)/f'(r).$$

We have used that f and f' are continous and $f'(r) \neq 0$ to take the limit. We may substract r from both sides and obtain $f(r) = 0$. ∎

2 Further examples

Example. Find the largest interval of the form (a, ∞) so that $f(x) = xe^x$ is one to one on this interval.

If g is the inverse function to f on the interval above, find $g(2)$.

Solution. In order for f to be one to one it must be monotone. If we compute $f'(x) = (x + 1)e^x$, we see that f will be increasing on the interval $(-1, \infty)$. Thus f is one to one there.

If $b = g(2)$, then b will solve the equation $h(b) = be^b - 2 = 0$. This equation can be solved by Newton's method. Thus $x_1 = 1$ is a reasonable starting point and we can use the iteration $x_{n+1} = x_n - h(x_n)/h'(x_n) = x_n - (x_n e^{x_n} - 2)/((x_n + 1)e^{x_n})$ to find successive approximations to solution of $h(b) = 0$.

Newton does the usual spectacular job:

n	x_n
1	1
2	0.86788
3	0.85278
4	0.85261

Substituting, we can verify that $0.85621e^{0.85621} \approx 2$. ∎

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