1 Lecture 37 and 38: The fundamental theorems of calculus.

- The fundamental theorems of calculus.
- Evaluating definite integrals.
- The indefinite integral-a new name for anti-derivative.
- Differentiating integrals.

Today we provide the connection between the two main ideas of the course. The integral and the derivative.

Theorem 1 *(FTC I) Suppose f is a continuous function on $[a, b]$. If F is an anti-derivative of f, then*

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

(FTC II) Assume f is continuous on an open interval I and a is in I. Then the area function

$$A(x) = \int_a^x f(t)\, dt$$

is an anti-derivative of f and thus $A' = f$.

Example. Compute

$$\frac{d}{dx} \int_1^x \frac{1}{t}\, dt.$$

Compute

$$\int_0^3 x^3\, dx.$$

Proof. An idea of the proofs.

FTC I: We let F be an anti-derivative of f and let $P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$. We will express the change of F, $F(b) - F(a)$, as a Riemann sum for this partition. Letting the size of the largest interval in the partition tend to zero, we obtain the integral is equal to the change in F.

We begin by writing

$$F(b) - F(a) = F(x_n) - F(x_{n-1}) + F(x_{n-1}) - \ldots + F(x_i) - F(x_{i-1}) + \ldots + F(x_1) - F(x_0).$$

We recall that F is an anti-derivative of f and apply the mean value theorem on each interval $[x_{i-1}, x_i]$ and find a value c_i so that $F(x_i) - F(x_{i-1}) = f(c_i)(x_i - x_{i-1})$. Thus, we have

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}).$$

Since the right-hand side is a Riemann sum for the integral, we may let the width of the largest subinterval tend to zero and obtain

$$F(b) - F(a) = \int_a^b f(s)\,ds.$$

FTC II:

Write

$$\frac{A(x+h) - A(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

We will show

$$\lim_{h\to 0^+} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x).$$

The reader should write out a similar argument for the limit from the left.

If f is continuous, then f has maximum and minimum values M_h and m_h on the interval $[x, x+h]$. Using the order property of the integral,

$$m_h \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le M_h.$$

As h tends to 0, we have $\lim_{h\to 0^+} M_h = \lim_{h\to 0^+} m_h = f(x)$ since f is continuous. It follows that

$$\lim_{h\to 0^+} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x).$$

∎

1.1 Indefinite integrals.

We use the symbol

$$\int f(x)\,dx$$

to denote the indefinite integral or anti-derivative of f.

The indefinite integral is a function. The definite integral is a number. According FTC I, we can find the (numerical) value of a definite integral by evaluating the indefinite integral at the endpoints of the integral. Since this procedure happens so often, we have a special notation for this evaluation.

$$F(x)\big|_{x=a}^{b} = F(b) - F(a).$$

Example. Find

$$xa\big|_{x=a}^{b} \qquad \text{and} \qquad xa\big|_{a=x}^{y}$$

Solution.

$$ba - a^2 \qquad xy - x^2$$

∎

According to FTC II, anti-derivatives exist provided f is continuous.

The box on page 351 should be memorized. (In fact, you should already have memorized this information when we studied derivatives in Chapter 3 and when we studied anti-derivatives in Chapter 4.)

Example. Verify

$$\int x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2).$$

Solution. According to the definition of anti-derivative, we need to see if

$$\frac{d}{dx}\frac{1}{2}\sin(x^2) = x\cos(x^2).$$

This holds, by the chain rule. ∎

1.2 Computing integrals.

The main use of FTC I is to simplify the evaluation of integrals.

We give a few examples.

Example. a) Compute

$$\int_0^\pi \sin(x)\,dx.$$

b) Compute

$$\int_1^4 \frac{2x^2+1}{\sqrt{x}}\,dx.$$

Solution. a) Since $\frac{d}{dx}(-\cos(x)) = \sin(x)$, we have $-\cos(x)$ is an anti-derivative of $\sin(x)$. Using the second part of the fundamental theorem of calculus gives,

$$\int_0^\pi \sin(x)\,dx = -\cos(x)|_{x=0}^{\pi} = 2.$$

b) We first find an anti-derivative. As the indefinite integral is linear, we write

$$\int \frac{2x^2+1}{\sqrt{x}}\,dx = \int 2x^{3/2} + x^{-1/2}\,dx \quad = \quad 2\int x^{3/2}\,dx + \int x^{-1/2}\,dx = \frac{4}{5}x^{5/2} + 2x^{1/2} + C.$$

With this anti-derivative, we may then use FTC I to find

$$\begin{aligned}
\int_1^4 \frac{2x^2+1}{\sqrt{x}}\,dx &= \frac{4}{5}x^{5/2} + 2x^{1/2}\Big|_{x=1}^{4} \\
&= \frac{4}{5}4^{5/2} + 24^{1/2} - (\frac{4}{5} + 2) \\
&= 128/5 + 20/5 - (4/5 + 10/5) \\
&= 134/5.
\end{aligned}$$

∎

Example. Find

$$\int_0^{\sqrt{\pi}} 2x\cos(x^2)\,dx.$$

Solution. We recognize that $\sin(x^2)$ is an anti-derivative of $2x\cos(x^2)$,

$$\int 2x\cos(x^2)\,dx = \sin(x^2) + C.$$

Thus,

$$\int_0^{\sqrt{\pi}} 2x\cos(x^2)\,dx = \sin(x^2)\Big|_{x=0}^{\sqrt{\pi}} = 0 - 0.$$

∎

Here, is a more involved example that illustrates the progress we have made.

Example. Find

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \sin(k/n).$$

Solution. We recognize that

$$\frac{1}{n}\sum_{k=1}^{n} \sin(k/n)$$

is a Riemann sum for an integral. The points x_k, $k = 0, \ldots, n$ divide the interval $[0, 1]$ into n equal sub-intervals of length $1/n$. Thus, we may write the limit as an integral

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \sin(k/n) = \int_0^1 \sin(x)\,dx.$$

To evaluate the resulting integral, we use FTC I. An anti-derivative of $\sin(x)$ is $-\cos(x)$, thus

$$\int_0^1 \sin(x)\,dx = -\cos(x)|_{x=0}^{1} = 1 - \cos(1).$$

∎

1.3 Differentiating integrals.

FTC II shows that any continuous function has an anti-derivative and can be used to find the derivatives of integrals.

Example. Find

$$\frac{d}{dx}\int_0^x \sin(t^2)\,dt \qquad L'(x)\text{if } L(x) = \int_1^x \frac{1}{t}\,dt \qquad \frac{d}{dx}\int_{x^2}^x \sin(t^2)\,dt$$

Is the function $L(x) = \int_1^x \frac{1}{t}\,dt$ increasing or decreasing? Is the graph of L concave up or concave down?

Solution. The first one is a straightforward application of the second part of the fundamental theorem. The function $\sin(x^2)$ is continuous everywhere and thus we have

$$\frac{d}{dx}\int_0^x \sin(t^2)\,dt = \sin(x^2).$$

The second one is also straightforward,

$$\frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}, \qquad x > 0.$$

Taking another derivative, we find that

$$\frac{d^2}{dx^2}\int_1^x \frac{1}{t}\,dt = -1/x^2.$$

Thus this function is concave down for $x > 0$.

Of course we can also use FTC I to see that $\int_1^x \frac{1}{t}\,dt = \ln(x) - \ln(1)$ and then apply the differentiation rules to compute the derivative. Note that we could not use this approach in the first example since we do not know an anti-derivative for $\sin(x^2)$.

Finally, the third one requires us to use the properties of the integral to put it in a form where we can use FTC II. We can write

$$\int_{x^2}^x \sin(t^2)\,dt = \int_{x^2}^0 \sin(t^2)\,dt + \int_0^x \sin(t^2)\,dt = -\int_0^{x^2} \sin(t^2)\,dt + \int_0^x \sin(t^2)\,dt.$$

Now applying FTC II and using the chain rule for the first integral gives

$$\frac{d}{dx}(-\int_0^{x^2} \sin(t^2)\,dt + \int_0^x \sin(t^2)\,dt) = -2x\sin(x^4) + \sin(x^2).$$

∎

Our second example shows that it is necessary to assume that f is continuous in FTC II.

Example. Let f be the function given by

$$f(x) = \begin{cases} 0, & x < 2 \\ 1, & x \geq 2 \end{cases}$$

Find $F(x) = \int_0^x f(x)\,dx$ and determine where F is differentiable.

Solution. We have that the integral is given by

$$F(x) = \begin{cases} 0, & x < 2 \\ (x-2), & x \geq 0 \end{cases}$$

It is pretty clear that F is differentiable everywhere except at 2. At 2, we can compute the left and right limits of the difference quotient and find

$$\lim_{h\to 0^-} \frac{F(2+h) - F(2)}{h} = 0 \qquad \lim_{h\to 0^+} \frac{F(2+h) - F(2)}{h} = 1.$$

Thus $F'(2)$ does not exist. ∎

1.4 The net change theorem

Since F is always an anti-derivative of F', one consequence of part I of the fundamental theorem of calculus is that if F' is continuous on the interval $[a, b]$, then

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

This helps us to understand some common physical interpretations of the integral.

Example. An object falls with constant acceleration g, at $t = 1$ its height is h_1 and its velocity is v_1. Find its position at all times.

Solution. By the net change theorem,

$$v(t) - v(1) = \int_1^t g\,ds = g(t-1).$$

Thus $v(t) = g(t-1) + v_1$. Applying the net change theorem again we have the height at time time t, $h(t)$ is

$$h(t) - h(1) = \int_1^t g(s-1) + v_1\,ds = \frac{1}{2}g(s-1)^2|_{s=1}^t + v_1 s|_{s=1}^t = \frac{1}{2}g(t-1)^2 + v_1(t-1).$$

Thus

$$h(t) = \frac{1}{2}g(t-1)^2 + v_1(t-1) + h_1.$$

∎

Note this gives a different version of the equations for a falling object.

To give a less familiar example, suppose we have a rope whose thickness varies along its length. Fix one end of the rope to measure from and let $m(x)$ denote the mass in kilograms of the rope from 0 to x meters along the rope. If we take the derivative, $\frac{dm}{dx} = \lim_{h\to 0}(m(x+h) - m(x))/h$, then this represents mass per unit length (or linear density) of the rope near x and the units are kilograms/meter. If we integrate this linear density and observe that $m(0) = 0$, then we recover the mass

$$m(x) = \int_0^x \frac{dm}{dx}\,dx.$$

This is another example of the net change theorem.