# 1 Lecture 40: Exponential growth and decay

- A model for exponential growth and decay
- Fitting our solution to data, doubling time and half-life
- Example: Population growth
- Example: Carbon dating

### 1.1 A model for population growth

A simple model for a population is to assume that the rate of change of the population is directly proportional to the total population. If y is the total population and b is the birth rate or the fraction of the population that give birth each unit of time, then the rate of change due to births is by. Similarly if d is the death rate, then dy is the number of deaths per year. The rate of change of the population y with respect to time is by - dy. We can express this using a derivative as

$$\frac{dy}{dt} = ky \tag{1}$$

where the constant k = b - d. Since y is positive, we have that y is increasing if k > 0and y is decreasing if k < 0. It is easy to see that the family of functions  $y(t) = P_0 e^{kt}$ are solutions of (1). Here,  $P_0$  is a constant and each choice of  $P_0$  gives a different solution of (1). In fact, these are the only solutions.

**Theorem 2** If y solves (1) on an interval I, then there is a constant  $P_0$  so that  $y(t) = P_0 e^{kt}$  on I.

*Proof.* If we want to show  $y(t) = P_0 e^{kt}$ , then we expect that  $e^{-kt}y(t)$  is constant. One way to show a function is constant is to show the derivative is zero. We consider the function  $f(t) = e^{-kt}y(t)$  and differentiate f using the product rule,

$$f'(t) = (-k)e^{-kt}y(t) + e^{-kt}y'(t) = -ke^{-kt}y + e^{-kt}ky = 0.$$

We have used (1) for the second equality. Since f' = 0 on an interval, f is constant. If we call the constant  $P_0$ , then we have  $f(t) = P_0 e^{kt}$ .

### **1.2** Model fitting

If a function y is given by  $y(t) = P_0 e^{kt}$  and k > 0 we say that y grows exponentially. If k < 0, then we say that y decays exponentially. In this case we will often replace k by -k and write  $y(t) = P_0 e^{-kt}$ . Thus the constant k is positive. Once we know that we have exponential growth and decay, we need two additional bits of data to determine the constant k and the value of  $P_0$ . Note we have  $y(0) = P_0$  so  $P_0$  is the initial value of y. *Example.* Suppose that y obeys (1), y(1) = 2 and y(2) = 5. Find y(t).

Solution. We know that  $y(t) = P_0 e^{kt}$ . The given information tells us that

$$P_0 e^k = 2, \qquad P_0 e^{2k} = 5.$$

To find k, we may divide these equations and find

$$\frac{P_0 e^{2k}}{P_0 e^k} = \frac{5}{2}$$

Taking the natural logarithm of both sides, we find  $k \ln(e) = \ln(5/2)$ . Thus,  $y(t) = P_0 e^{t \ln(5/2)}$ . Substituting t = 1, we find  $P_0 e^{\ln(5/2)} = 2$  or  $P_0 = 4/5$ . Summarizing,

$$y(t) = \frac{4}{5}e^{t\ln(5/2)} = \frac{4}{5}\left(\frac{5}{2}\right)^t.$$

One of the important features of exponential growth is the existence of a time T during which the population doubles, i.e. that y(t + T) = 2y(t). To see that this doubling property is independent of t, we consider

$$\frac{y(t+T)}{y(t)} = \frac{P_0 e^{k(t+T)}}{P_0 e^{kt}} = e^{kT}.$$

Thus, to find the doubling time we need to solve  $e^{kT} = 2$  for T.

*Example.* If  $f(t) = 100e^{0.3t}$ , find the doubling time. Can you find the tripling time?

Solution. We have

$$\frac{f(t+T)}{f(t)} = e^{0.3T}.$$

We solve the equation  $e^{0.3T} = 2$  to find  $T = \ln(2)/0.3$ .

The same argument gives that the tripling time is  $\ln(3)/0.3$ .

In the case of exponential decay, the corresponding notion is half-life. This is the time T so that  $y(t+T) = \frac{1}{2}y(t)$ .

#### **1.3** Example: Exponential growth

*Example.* Suppose that a population grows at a rate of 3% per day. If the initial population is 100, when will the population reach 1000. What is the doubling time?

Solution. Let y(t) denote the population at time t where t is measured in days. If y(t) is the population at time t, we know that y' = 0.03y and y(0) = 100. Thus,  $y(t) = 100e^{0.03t}$ . We are asked to find the time T when y(T) = 1000. Thus we want  $y(T) = 100e^{0.03T} = 1000$ . Thus we need  $e^{0.03T} = 10$  or  $T = \ln(10)/0.03 \approx 76.753$  days.

To find the doubling time, we solve

$$100e^{0.03(t+T)} = 2 \cdot 100e^{0.03t}$$

for T to find  $e^{0.03T} = 2$  or  $T = \ln(2)/0.03 \approx 23.1$  days.

## 1.4 Example: Carbon dating

The carbon in the atmosphere includes two isotopes  $C_{14}$  and  $C_{12}$  and the ratio of these isotopes in living plants and animals is roughly the same as in the atmosphere. When the organism dies, the  $C_{14}$  starts to decay. If R(t) represents the ratio of  $C_{14}$ to  $C_{12}$  at a time t years after the organism's death, we find that  $R(t) = R_a e^{-kt}$  where  $R_a$  is ratio of  $C_{14}$  to  $C_{12}$  in the atmosphere. The half-life of C14 is approximately 5730 years.

*Example.* Suppose that in a sample of wood, the ratio of C14 to  $C_{12}$  is 23% of the ratio in the atmosphere. How long ago was the wood in a living tree?

Solution. Let R(t) the ratio of  $C_{14}$  to  $C_{12}$  at a time t years after the tree dies. As  $R(t) = R_a e^{-kt}$ , we want to find the time T so that  $R(T) = 0.23R_a$ .

Before we can do this, we need to find k. We use that the half-life is 5730 years to find k. Since  $e^{-k5730} = \frac{1}{2}$ , we have that  $-k = \ln(1/2)/5730$  of  $k = \ln(2)/5730 \approx 1.21 \cdot 10^{-4}$ . Thus, if  $e^{-kT} = 0.23$ , we may solve for T and find that  $T = \ln(0.23)/(-k) = \ln(0.23)5730/\ln(2) \approx 12,149$  years.

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