1 Lecture 08: The squeeze theorem

- The squeeze theorem
- The limit of $\sin(x)/x$
- Related trig limits

1.1 The squeeze theorem

Example. Is the function g defined by

$$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

continuous?

Solution. If $x \neq 0$, then $\sin(1/x)$ is a composition of continuous function and thus $x^2 \sin(1/x)$ is a product of continuous function and hence continuous.

If $x = 0$, we need to have that $\lim_{x \to 0} g(x) = g(0) = 0$ in order for g to satisfy the definition of continuity. Recalling that $\sin(1/x)$ oscilates between $-1 \leq x \leq 1$, we have that

$$-x^2 \leq g(x) \leq x^2$$

and since $\lim_{x \to 0} x^2 = \lim_{x \to 0} -x^2 = 0$, the theorem below tells us we have $\lim_{x \to 0} g(x) = 0$. ∎

Theorem 1 (The squeeze theorem) *If f, g, and h are functions and for all x in an open interval containing c, but perhaps not at c, we have*

$$f(x) \leq g(x) \leq h(x)$$

and

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L,$$

then

$$\lim_{x \to c} g(x) = L.$$

We will not give a proof but it should be intuitive that if g is trapped between two functions that approach the limit L, then g also approaches that limit.

Example. Suppose that for all real numbers x, we have

$$a \leq f(x) \leq 1 + x^2.$$

There is exactly one choice for a, c, and L so that

$$\lim_{x \to c} f(x) = L.$$

Find a, c, and L.

Solution. For the squeeze theorem to apply, we need the graphs of $y = 1$ and $y = 1 + x^2$ to touch at one point. This means the equation $1 + x^2 = a$ will have exactly one solution. This will happen only if $a = 1$ and the solution is $x = 0$. Thus we have $1 \le f(x) \le 1 + x^2$ for all x and the squeeze theorem tells us that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} 1 = \lim_{x\to 0}(1 + x^2) = 1.$$

∎

1.2 The limit of $\sin(x)/x$

We consider the limit

$$\lim_{x\to 0} \frac{\sin(x)}{x}.$$

The quotient rule for limits does not apply since the limit of the denominator is 0. Unlike our previous limits, we cannot simplify to obtain a function where we can use the direct substitution rule or another rule. Instead, we will use the squeeze theorem.

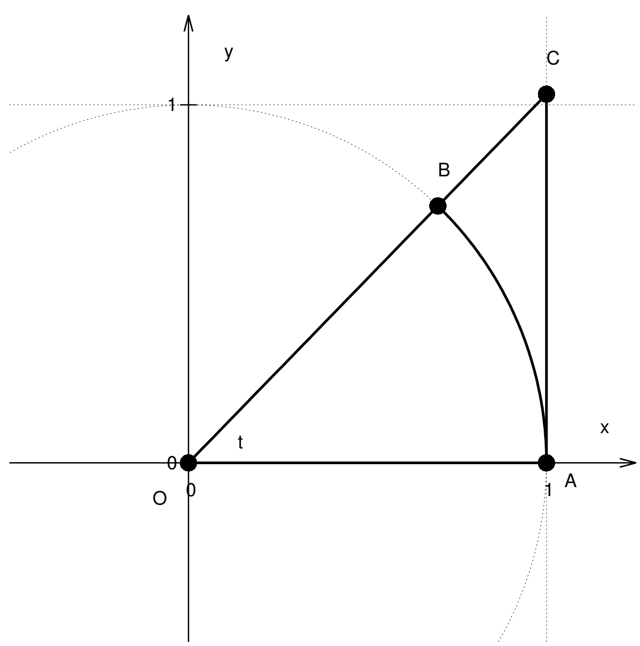
Theorem 2

$$\lim_{t\to 0} \frac{\sin(t)}{t}.$$

Proof. We start by observing that $\sin(-t)/(-t) = \sin(t)/t$, so it suffices to consider $\lim_{t\to 0^+} \sin(t)/t$.

In the figure below we draw an angle t with $0 < t < \pi/2$ and observe that we have the inequalities

$$\text{Area triangle } OAB \le \text{Area sector } OAB \le \text{Area triangle } OAC.$$



We have

$$\begin{aligned}
\text{Area triangle } OAB &= \frac{1}{2}\sin(t) \\
\text{Area sector } OAB &= \frac{1}{2}t \\
\text{Area triangle } OAC &= \frac{1}{2}\tan(t)
\end{aligned}$$

Thus we have

$$\frac{1}{2}\sin(t) \le t/2 \le \frac{1}{2}\tan(t).$$

Since $t > 0$, we can rearrange to obtain

$$\cos(t) \le \frac{\sin(t)}{t} \le 1. \tag{3}$$

and since $\sin(-t)/(-t) = \sin(t)/t$, we also have (3) if $0 < |t| < \pi/2$. Since $\lim_{t\to 0}\cos(t) = 1$, the squeeze theorem implies

$$\lim_{t\to 0}\frac{\sin(t)}{t} = 1.$$

∎

1.3 Some consequences

Using this limit, we can find several related limits.

The first one will be used in the next chapter.

Example. Find the limit

$$\lim_{x\to 0}\frac{1-\cos(x)}{x}.$$

Solution. We note that since the limit of the denominator is zero, we cannot use the quotient rule for limits. However, if we multiply and divide by $1+\cos(x)$ and use the identity $\sin^2(x)+\cos^2(x) = 1$, we have

$$\frac{1-\cos(x)}{x} = \frac{(1-\cos(x))(1+\cos(x))}{x(1+\cos(x))} = \frac{\sin^2(x)}{x}.$$

Thus, we may use the rule for a limit of a product,

$$\lim_{x\to 0}\frac{1-\cos(x)}{x} = \lim_{x\to 0}\frac{\sin^2(x)}{x} = \lim_{x\to 0}\sin(x)\lim_{x\to 0}\frac{\sin(x)}{x} = 0.$$

∎

Below are a few more to try

1. $\lim_{t\to 0} \frac{\sin(2t)}{t}$
2. $\lim_{t\to 0} \frac{\sin(2t)}{\sin(3t)}$
3. $\lim_{t\to 0} \frac{1-\cos(t)}{t^2}$

Solution. To find the limit $\lim_{t\to 0} \sin(2t)/t$, we begin by multiplying and dividing by 2 to obtain

$$\frac{\sin(2t)}{t} = \frac{2\sin(2t)}{2t}.$$

Now if t is close to zero, then $2t$ will also be close to zero. Thus,

$$\lim_{t\to 0} \frac{\sin(2t)}{t} = \lim_{t\to 0} \frac{2\sin(2t)}{2t} = 2\lim_{u\to 0} \frac{\sin(u)}{u} = 2.$$

∎

Example. Suppose that for all real numbers x, we have

$$a \leq f(x) \leq x^2 + 6x$$

There is exactly one value of a for which we can use the squeeze theorem to evaluate the limit

$$\lim_{x\to c} f(x) = L.$$

Find a, c, and L.

Solution. In order for the squeeze theorem to apply, we need for the equation

$$x^2 + 6x = a \qquad \text{or} \qquad x^2 + 6x - a = 0$$

to have exactly one solution. From the quadratic formula the solutions are

$$\frac{-6 \pm \sqrt{36 + 4a}}{2}.$$

This will give us one solution when the discriminant, $36+4a$, is zero or if $36+4a = 0$. Solving this question gives $a = -9$. We have the inequality $x^2 + 6x \geq -9$ for all x with equality only at $x = -3$. Since $\lim_{x\to -3} x^2 + 6x = \lim_{x\to -3} -9 = -9$, the squeeze theorem will imply that

$$\lim_{x\to -3} f(x) = -9.$$

∎