1 Lecture 12: The derivative.

1.1 Outline

- Definition of the derivative at a point and differentiability
- Derivative of linear function
- Definition of a tangent line
- Example: finding a tangent line
- Derivatives of powers
- Example: a function which is not differentiable
- Example: Derivative of x^3

1.2 The derivative

Definition. Given a function f, which is defined in an interval containing a, we may define the *derivative* of f at a by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exists. If the $f'(a)$ exists, we say that f is *differentiable* at a.

The expression $\frac{f(x)-f(a)}{x-a}$ gives an average rate of change and the limit defines the rate of change of f at a point a. We call the expression $\frac{f(x)-f(a)}{x-a}$ the *difference quotient* of f at a.

An equivalent definition that is sometimes useful is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

You should try both formulations and learn which is easier for you to use. We will see problems where it is helpful to know both formulations.

We find the derivative of some simple functions.

1.3 Example

Example. Find the derivative of a linear function $f(x) = mx + b$.

Solution. We write out the difference quotient

$$\frac{f(x)-f(a)}{x-a} = \frac{mx+b-(ma+b)}{x-a} = m, \qquad x \neq a.$$

The limit of the difference quotient is the slope m of the linear function. Thus $f'(x) = m$ and the domain of f' is all real numbers. ∎

Definition. The *tangent line* to the graph of a function f at a point a is the line which passes through $(a, f(a))$ and has slope the derivative of f at a, $f'(a)$. In point slope form the equation of the tangent line is

$$y - f(a) = f'(a)(x-a).$$

Example. Find the tangent line to the parabola $y = x^2$ at $x = 2$. Check your answer by graphing the parabola and the tangent line.

Solution. We first need to find the derivative of $f(x) = x^2$ at 2. Using the definition, this is

$$\lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h\to 2} 4 + h = 4.$$

Thus the tangent line passes through (2,4) and has slope 4. The equation is

$$y - 4 = 4(x-2).$$

Simplifying to put this in point-slope form gives

$$y = 4x - 4.$$

Be sure to check. ∎

Example. Find the derivative of the function $f(x) = x^n$ at a number a where $n = 1, 2, 3 \ldots$ is a positive integer.

Solution. We will use the factoring pattern

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \ldots + a^{n-1}) = (x-a)\sum_{k=0}^{n-1} x^{n-1-k}a^k.$$

This may be checked by multiplying out the left-hand side.

With this we may simplify the difference quotient

$$\frac{x^n - a^n}{x-a} = (x^{n-1} + x^{n-2}a + \ldots + a^{n-1}), x \neq a.$$

Since the right-hand side is a polynomial in x and hence continuous, we may use the direct substitution rule to find

$$\lim_{x\to a}(x^{n-1} + x^{n-2}a + \ldots + a^{n-1}) = na^{n-1}.$$

∎

Remark. Even though we have not proved this rule for all cases yet, it is true that if $f(x) = x^r$, then $f'(x) = rx^{r-1}$ for any real number r. We will use this rule in further problems. ∎

Example. Can you find the derivative of $f(x) = |x|$ at $x = 0$? At other values of x?

Solution. If we write down the difference quotient we find that

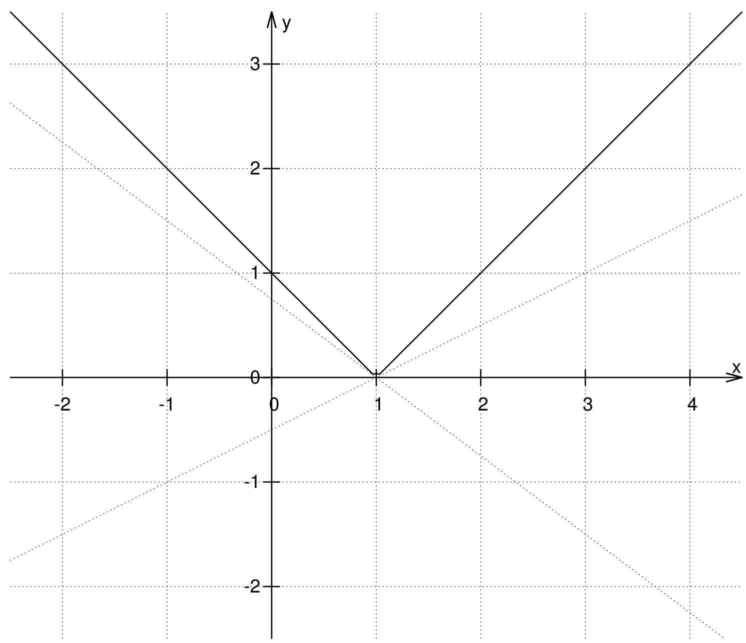
$$\frac{f(x) - f(0)}{x} = \frac{|x|}{x}.$$

We have computed the one-sided limits for this function and we know

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1 \qquad \lim_{x \to 0^-} \frac{|x|}{x} = -1.$$

Since the two one-sided limits have different values, the limit of the difference quotient at 0 does not exist. The function $f(x) = |x|$ is not differentiable at 0.

An examination of the graph helps to see that there is no tangent line at 0.



∎

Example. Find the derivative of $f(x) = x^3$ at a general point b using the difference quotient in the form $((b + h)^3 - b^3)/h$.

1.4 Approximating the derivative

Sometimes, we may not have a formula for a function and want to approximate the derivative. We may do this by computing difference quotients

$$\frac{f(x) - f(a)}{x - a} \qquad \text{or} \frac{f(a+h) - f(a)}{h}$$

where h is close to zero or x is close to a.

Example. Estimate the derivative $f'(0)$ for $f(x) = \sin(x)$.

Solution. If we consider

$$\frac{f(0+h) - f(0)}{h} = \frac{\sin(h)}{h},$$

for small values of h, we find

h	$\sin(h)/h$
0.1	0.9983
0.02	0.99993

From this table it is reasonable to guess that $f'(0) = 1$. ∎

1.5 Exercises

1. Find the derivative of $g(x) = 1/x$.
2. Compute the derivative of $g(x) = 1/(3 - 2x)$ using the definition.
3. Estimate the derivative of e^x at $x = 0$.
4. We can define the symmetric derivative of f, f^s by

$$f^s(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

   (a) If f is differentiable at x, show that $f^s(x)$ exists and $f^s(x) = f'(x)$.
   (b) Can you find a function f so that $f^s(0)$ exists, but $f'(0)$ does not exist?

5. Suppose that $f'(x)$ exists and consider the limit,

$$g(x) = \lim_{h \to 0} \frac{f(x+ah) - f(x+bh)}{h}.$$

   What is the relation between $f'(x)$ and $g(x)$?