1 Lecture 13: The derivative as a function.

1.1 Outline

- Definition of the derivative as a function. definitions of differentiability.
- Power rule, derivative the exponential function
- Derivative of a sum and a multiple
- Differentiability implies continuity.
- Example: Finding a derivative.

1.2 The derivative

Definition. Given a function f, we may define a new function f', which we call *the derivative of* f by the rule that $f'(x)$ is the derivative at x.

Recalling the definition of the derivative at a point, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

provided the limit exists. The domain of f' is exactly the set of points where f is differentiable.

We will sometimes use a different notation for the derivative, d/dx, instead of $'$. The symbol f' and the Leibniz notation df/dx both denote the same function,

$$\frac{df}{dx} = f'.$$

The Leibniz notation is particular convenient for functions that are given by a formula, but have no name. For example, in the last class we showed that

$$\frac{d}{dx} x^n = nx^{n-1}.$$

1.3 Some formulae

We have two important differentiation formulas:

$$\frac{d}{dx} x^n = nx^{n-1}, n = 1, 2, 3, \ldots$$

and

$$\frac{d}{dx} e^x = e^x.$$

The first was proved in our previous lecture.

Computing the second derivative is more difficult. Let b^x be an exponential function to an arbitrary base, $b > 0$. From the properties of b^x, we have

$$\frac{b^{x+h} - b^x}{h} = \frac{b^x b^h - b^x}{h} = \frac{b^h - 1}{h} b^x.$$

It is true that the limit $\lim_{h\to 0} \frac{b^h-1}{h} = m(b)$ exists. We will assume this fact. The number e is special because it is the only number where this limit is 1,

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1. \tag{1}$$

The property (1) can be used to define e and helps to explain the special role of e in mathematics. Thus we have that the function e^x is its own derivative,

$$\frac{d}{dx} e^x = e^x.$$

1.4 Derivatives of sums

Theorem 1 *If f and g are differentiable at x and c is a real number, then $f + g$ and cf are differentiable at x and*

$$(f + g)'(x) = f'(x) + g'(x) \qquad \text{and} \qquad (cf)'(x) = cf'(x).$$

Proof. We consider the difference quotient for $f + g$ and write as

$$\frac{(f + g)(y) - (f + g)(x)}{x - y} = \frac{f(y) - f(x)}{x - y} + \frac{g(y) - g(x)}{x - y}.$$

Since we know each of the difference quotients on the right has a limit, we may use the sum rule for limits

$$\lim_{y\to x} \frac{(f + g)(y) - (f + g)(x)}{x - y} = \lim_{y\to x} \frac{f(y) - f(x)}{x - y} + \lim_{y\to x} \frac{g(y) - g(x)}{x - y}.$$

Thus $(f + g)'(x) = f'(x) + g'(x)$.

We omit the proof of the second one. ∎

With these rules and the power rule, we can now find the derivative of every polynomial.

Example. Find the derivative of $f(x) = 3x^4 + 4x^3$.

Solution. $12(x^3 + x^2)$. ∎

Example. Find the derivative of $f(x) = \dfrac{x^3 + 2x^4}{x}$.

Solution. We do not know how to differentiate this function as it is written. However, f simplifies to $f(x) = x^2 + 2x^3$ and now the derivative is $2x + 6x^2$. ∎

1.5 Differentiability and continuity.

Theorem 2 *If f is differentiable at x, then f is continuous at x.*

Proof. To show f is continuous at x, we will show that

$$\lim_{y \to x} (f(y) - f(x)) = 0.$$

We can use the product rule for limits and the differentiability of f to see that

$$\lim_{y \to x} (f(y) - f(x)) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}(y - x)) = f'(x) \cdot 0 = 0.$$

∎

Example. Show that the function

$$f(x) = \begin{cases} x, & x < 1 \\ 2, & x \geq 1 \end{cases}$$

is not differentiable at 1.

Solution. If the function where differentiable, it would be continuous at 1. Since it is not continuous at 1, it cannot be differentiable there. ∎

1.6 Examples

Example. Let $f(x) = 1/x$. Find all values x where the slope of the tangent line at x is 4. Find all values x where the slope of the tangent line is -4.

Find all tangent lines to the graph of f which are parallel to the line $y = -4x$.

Solution. We may write $f(x) = 1/x = x^{-1}$ and find the derivative $f'(x) = -x^{-2}$. We see that $f'(x) < 0$ and thus there is no point where the tangent line has slope 4. To find points where the slope of the tangent line is -4, we need to solve $f'(x) == -1/x^2 = -4$. The solutions are $x = \pm 1/2$. Thus there are two tangent lines to the graph with slope -4. They are the line with slope -4 which pass through $(1/2, 2)$ and the line with slope -4 with slope $(-1/2, -2)$. The equations are

$$y - 2 = -4(x - 1/2) \qquad y + 2 = -4(x + 1/2).$$

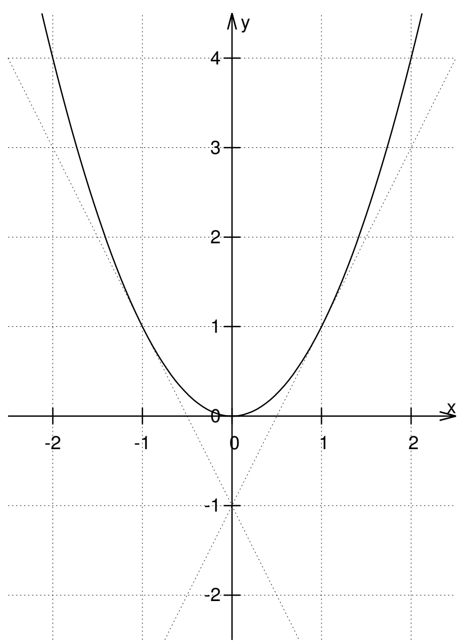
∎

Example. Can you find tangent lines to the graph $y = x^2$ which pass through $(0, -1)$.

Solution. The general tangent line to the graph of $f(x)$ at the point $(a, f(a))$ is $y - f(a) = f'(a)(x - a)$. If the point $(0, -1)$ is to lie on this line, we must have $-1 - f(a) = f'(a)(0 - a)$. In the case of $f(x) = x^2$ and $f'(x) = 2x$, this becomes

$$\begin{aligned} -1 - a^2 &= 2a(0 - a) \\ a^2 &= 1 \\ a &= \pm 1 \end{aligned}$$

Thus the lines are tangent to the graph of $f(x) = x^2$ at the points $(1, 1)$ and $(-1, 1)$. The equation of line through $(1, 1)$ with slope 2 is $y - 1 = 2(x - 1)$ or $y = 2x - 1$ and the line through $(-1, 1)$ with slope -2 is $y - 1 = -2(x + 1)$ or $y = -2x - 1$. A sketch serves to check our answer.



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Example. Sketch the graph of $\sin(x)$ and make a rough sketch of the graph of the derivative, $\sin'(x)$. Can you guess the derivative of sin?

Example. Find the derivative of $f(x) = \sqrt{x}$.

Solution. For $f(x) = \sqrt{x}$, we look at the difference quotient

$$\begin{aligned}\frac{f(y)-f(x)}{x-y} &= \frac{\sqrt{y}-\sqrt{x}}{y-x} \\ &= \frac{\sqrt{y}-\sqrt{x}}{y-x}\frac{\sqrt{y}+\sqrt{x}}{\sqrt{y}+\sqrt{x}} \\ &= \frac{y-x}{y-x}\frac{1}{\sqrt{x}+\sqrt{y}} = \frac{1}{\sqrt{y}+\sqrt{x}}\end{aligned}$$

As long as $x > 0$, we may use the direct substitution rule to take the limit of the last expression and find

$$f'(x) = \lim_{y \to x} \frac{1}{\sqrt{y}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Since the limit exists for all $x > 0$, the derivative is $f'(x) = 1/(2\sqrt{x})$ with domain the interval $(0, \infty)$. ∎

1.7 Exercises

1. Find the derivatives
   (a) $2x^5 + 3x^2$
   (b) $3\sqrt{x} + 1/x$
   (c) $(3x)^2 + 3x$
   (d) $\sqrt{4x}$
   (e) $4x + e^x$
   (f) $\frac{5x^4+3x}{2x^3}$
2. Which of the following functions can we differentiate using only the rules introduced so far?
   (a) $4x^3 + 3x$
   (b) $\frac{x+3}{x}$
   (c) $\frac{x}{x+3}$
   (d) $(4x+1)^2$
   (e) e^{2x}
3. Find the tangent line to the parabola $y = x^2 + x$ at $x = 1$.
4. Find the tangent line to the curve $y = 1/x$ that passes through $(0, 2)$. Check your work by graphing.