1 Lecture 16: Higher derivatives

1.1 Outline

- Definition of higher-order derivatives
- Examples

1.2 Higher order derivatives

The acceleration is the derivative of the derivative of position. We call this a second derivative. We often write f'' for the derivative of f' and we call f'' the second derivative. We can define derivatives of any order by

$$f^{(0)} = f \qquad \text{and} \qquad f^{(n)} = f^{(n-1)'}$$

for the result of differentiating n-times. For derivatives up to order 2 or 3, we add a prime ($'$) for each derivative. But eventually we get tired of writing primes (and our readers get tired of counting) and we switch to one of the notations

$$f^{(n)} \qquad \text{or} \qquad \frac{d^n f}{dx^n}$$

Be careful to use the parentheses which help us to distinguish the nth-derivative $f^{(n)}$ from the nth power f^n.

1.3 Examples

Example. Find

$$\frac{d^2}{dx^2} x^2 e^x$$

If f is defined by $f(x) = e^x$, find $f^{(2014)}$.

Solution. The first derivative

$$\frac{d}{dx}(x^2 e^x) = x^2 e^x + 2xe^x = (x^2 + 2x)e^x$$

Differentiating again gives

$$\frac{d}{dx}(x^2 + 2x)e^x = (2x + 2)e^x + (x^2 + 2x)e^x = (x^2 + 4x + 2)e^x.$$

After spending all day computing 2014 derivatives, we find that $f^{(2014)}(x) = f^{(2011)}(x) = \ldots = f(x) = e^x$.

∎

Example. Can you find a formula for

$$\frac{d^n}{dx^n}x^n, \qquad \frac{d^{2n}}{dx^{2n}}x^n \qquad \frac{d^n}{dx^n}\frac{1}{x}?$$

Solution. Try a few examples and look for a pattern. ∎

Example. Find a polynomial $p(x)$ of degree 2 so that with $f(x) = e^x$, $p(0) = f(0)$, $p'(0) = f'(0)$, and $p''(0) = f''(0)$.

Compare the values of e^x and $p(x)$.

Can you suggest a better way to approximate e^x?

Solution. Since $f^{(k)}(x) = e^x$ for all k, we have $f^{(k)}(0) = 1$ for all $k = 0, 1, 2, \ldots$. If $p(x) = ax^2 + bx + c$, then we have

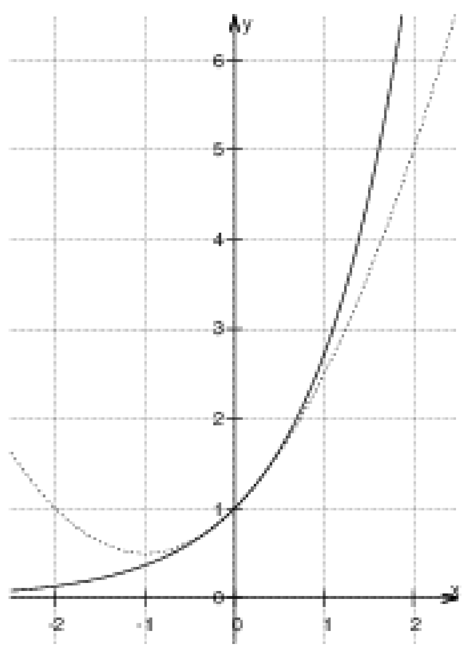
$$p(0) = c, \qquad p'(0) = b \qquad p''(0) = 2a.$$

Since we want $p(0) = f(0) = 1$, we have $c = 1$. If $p'(0) = 1$, then we must have $b = 1$ and if $p''(0) = 1$, we must have $2a = 1$ or $a = 1/2$. Thus $p(x) = \frac{1}{2}x^2 + x + 1$.

Trying a few values we find

x	e^x	$p(x)$
0	1	1
0.01	$1.010050167\ldots$	1.01005
0.2	$1.22140\ldots$	1.22
-0.2	$0.81873\ldots$	0.82

The graph below also indicates that the polynomial is a good approximation to the function near 0.



To do better we might look for a third degree polynomial with $p^{(k)}(0) = f^{(k)}(0)$ for $k = 0, 1, 2, 3$. ∎

1.4 Exercises

1. Compute the derivatives:

   (a) $\frac{d^3}{dx^3}(x^4 + x^2)$

   (b) $\frac{d^3}{dx^3}\frac{1}{x^2}$

   (c) $\frac{d^4}{dx^4}x^5$

   (d) $\frac{d^n}{dx^n}e^x$

   (e) $f^{(3)}(x)$ if $f(x) = e^{2x}$

   (f) $f^{(n)}(x)$ if $f(x) = e^{2x}$

2. Find the nth derivative of xe^x. Hint: Try a few and see if you can guess a pattern.

3. Compute
$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}.$$

October 1, 2015