1 Lecture 20: Implicit differentiation

1.1 Outline

- The technique of implicit differentiation
- Tangent lines to a circle
- Derivatives of inverse functions by implicit differentiation
- Examples

1.2 Implicit differentiation

Suppose we have two quantities or variables x and y that are related by an equation such as

$$x^2 + 2xy^2 + x^3y = xy.$$

If we know that $y = y(x)$ is a differentiable function of x, then we can differentiate this equation using our rules and solve the result to find y' or dy/dx. In this course, we will not learn conditions which guarantee that y is a differentiable function of x. This is a topic for a later course. This assumption is usually valid and the technique is very useful.

We begin with a simple example to practice the basic skill of differentiation as it is needed in implicit differentiation.

Example. Differentiate the expression below with respect to x. Assume that $y = y(x)$ is a function of x.

$$\frac{d}{dx}(x^2y^3)$$

Differentiate the expression below with respect to y. Assume that $x = x(y)$ is a function of y.

$$\frac{d}{dy}\sin(x + y^2).$$

Solution. For the first one, we begin by noting that $\frac{d}{dx}y^3 = 3y^2\frac{dy}{dx}$ by the chain rule. Then using the product rule,

$$\frac{d}{dx}x^2y^3 = 2xy^3 + 3x^2y^2\frac{dy}{dx}.$$

For the second one, we use the chain rule again to obtain

$$\frac{d}{dy}\sin(x + y^2) = (\frac{dx}{dy} + 2y)\cos(x + y^2).$$

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We show how to use this technique to find tangent lines to a circle.

Example. Consider the circle centered at the origin with radius 5 which is the set of points (x, y) which satisfy $x^2 + y^2 = 25$.

Find dy/dx on the circle.

Find the tangent lines at the points on the circle with x-coordinate 4.

Show that a tangent line to the circle is perpendicular to the radius at the point of tangency.

Solution. We imagine that $y = y(x)$ is a function of x in the equation defining the circle and differentiate both sides with respect to x.

$$\begin{aligned} \frac{d}{dx}(x^2 + y(x)^2) &= \frac{d}{dx}25 \\ 2x + 2y\frac{dy}{dx} = 0. \end{aligned}$$

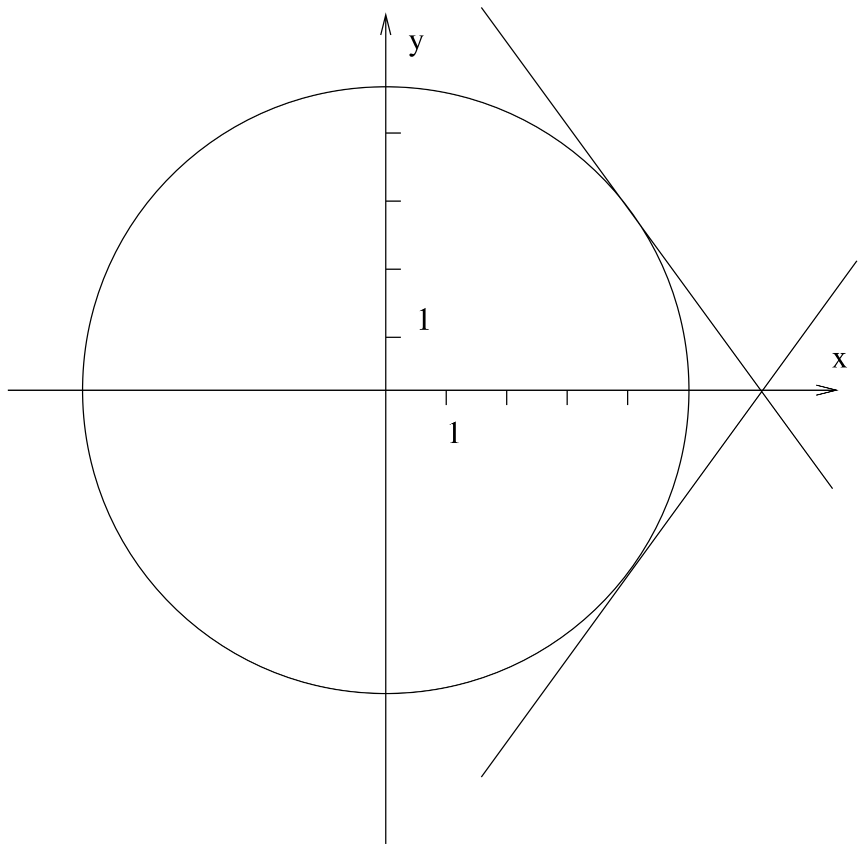
Observe that when we differentiated the term $y(x)^2$, we used the chain rule with $y(x)$ as the inside function. Next, we solve this equation for dy/dx to find

$$\frac{dy}{dx} = -x/y.$$

To find the tangent lines when the x-coordinate is 4, we solve $4^2 + y^2 = 25$ for y to find that $y = 3$ or -3. Thus there are two points where we need to find the tangent line. One passes through the point $(x, y) = (4, 3)$ and has slope $dy/dx = -4/3$. The second passes through $(x, y) = (4, -3)$ and has slope $dy/dx = 4/3$. The point slope forms of the equation are:

$$\begin{aligned} y - 3 &= \frac{-4}{3}(x - 4) \\ y + 3 &= \frac{4}{3}(x - 4) \end{aligned}$$

The following sketch shows the tangent lines and the circle and helps to check our answer.



Next, at a general point (x, y) on the circle the tangent line has slope $-x/y$ while the radius which is the line segment joining (x, y) to $(0, 0)$ has slope y/x. The product of these slopes is -1 and hence the lines are perpendicular. ∎

Exercise. We can also find tangent lines by solving the equation $x^2+y^2 = 25$ to give $y = \pm\sqrt{25 - x^2}$ and then using techniques we learned earlier.

Carry this out to check your answer to the previous problem.

Example. Find the second derivative y'' at the point $(3, 4)$ on the circle $x^2+y^2 = 25$.

Note that in this problem we use the notation y' for the derivative of y with respect to x, rather than the Leibniz notation, dy/dx.

Solution. We begin as before by differentiating $x^2 + y^2 = 25$ with respect to x and obtain

$$2x + 2yy' = 0. \tag{1}$$

As before, we have $y' = -x/y$ and we would like to differentiate again. It is probably simpler to differentiate (1) rather than $y' = -x/y$ to avoid using the quotient rule. Differentiating both sides of (1) with respect to x and using the product rule on the second term gives

$$2 + 2yy'' + 2(y')^2 = 0.$$

Solving for y'' gives

$$y'' = -(x + (y')^2)/y = -\frac{1}{y} - \frac{x^2}{y^3}.$$

In the second step we used that $y' = -x/y$. Now we may substitute the values $(x, y) = (3, 4)$ to obtain

$$y'' = -1/4 - 9/64 = -25/64.$$

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Example. Suppose that s and t are related by the equation $s^2 + te^{st} = 2$. Find ds/dt.

Solution. We assume that s is a function of t, $s(t)$, differentiate both sides of the equation defining the curve and group the terms involving ds/dt obtaining

$$\begin{aligned} \frac{d}{dt}(s^2 + te^{st}) &= \frac{d}{dt}2 \\ 2s\frac{ds}{dt} + e^{st} + t(s + t\frac{ds}{dt})e^{st} &= 0 \\ (2s + ste^{st})\frac{ds}{dt} + ste^{st} &= 0 \end{aligned}$$

We used the product rule and the chain rule to carry out the differentiation. Solving for ds/dt gives

$$\frac{ds}{dt} = \frac{-ste^{st}}{2s + ste^{st}}.$$

Note that even if the equation relating s and t, the equation for ds/dt is a linear equation and is easily solved. ∎

1.3 Derivatives of inverse functions

The technique of implicit differentiation can also be used to find the derivative of inverse functions. We illustrate this by finding the derivative of the function $\sin^{-1}(x)$.

Example. Find the derivative of the inverse sine function $\sin^{-1}$ or arcsin.

Solution. If $y = \sin^{-1}(x)$, then we have that

$$\sin(y) = x.$$

Differentiating equation with respect to x and recalling that $y = y(x)$ is a function of x gives that

$$y' \cos(y) = 1 \text{ or } y' = \frac{1}{\cos(y)}.$$

In order to simplify this last expression, we recall the pythagorean identity, $\sin^2(y) + \cos^2(y) = 1$ or $\cos(y) = \pm\sqrt{1-\cos^2(y)}$. Our definition of the $\sin^{-1}$ tells us that y is in the range $[-\pi/2, \pi/2]$ and thus that $\cos(y) \geq 0$. Thus we have $\cos(y) = \sqrt{1-\sin^2(y)} = \sqrt{1-\sin^2(\sin^{-1}(x))} = \sqrt{1-x^2}$. This gives the expected result that

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}.$$

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1.4 Additional examples

Example. Find the tangent line to the curve defined by $x^2 + 2y^2 = 2 + x^2y$ at the point $(x, y) = (3, 1)$.

Solution. The tangent line will go through the given point $(3, 1)$ thus the only thing we need to find is the slope, y'. We visualize that $y = y(x)$ is a function of x and differentiate both sides of the equation

$$(x^2 + 2y(x)^2)' = (2 + x^2y(x))'$$

where $'$ denotes the derivative with respect to x. We use the product and chain rules to conclude

$$2x + 4yy' = 0 + 2xy + x^2y'.$$

We solve this equation for y' and obtain

$$y'(4y - x^2) = 2xy - 2x \text{ or } y' = \frac{2xy - 2x}{4y - x^2}.$$

Substituting $(x, y) = (3, 1)$ gives

$$y' = \frac{6-6}{4-9} = 0.$$

Thus the tangent line is the line through $(3, 1)$ with slope 0 which gives

$$y = 1.$$

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In our last example, we will not use x and y. It is useful to remember that the technique of implicit differentiation can be used to find the rate of change between any two variables.

Example. Consider the quadratic equation

$$x^2 + 2x + c = 0.$$

a) Find the roots when $c = 0$.

b) Find the derivative of x with respect to c and for each root from part a) determine if the root increases or decreases as c increases.

c) Sketch the parabola $y = x^2 + x + c$ for $c = 0$ and check if your answer in part b) makes sense.

Solution. a) When $c = 0$, the equation $x^2 + 2x = 0$ factors as $x(x + 2) = 0$. The roots are $x = 0$ and $x = -2$.

b) We differentiate the equation with respect to c and find

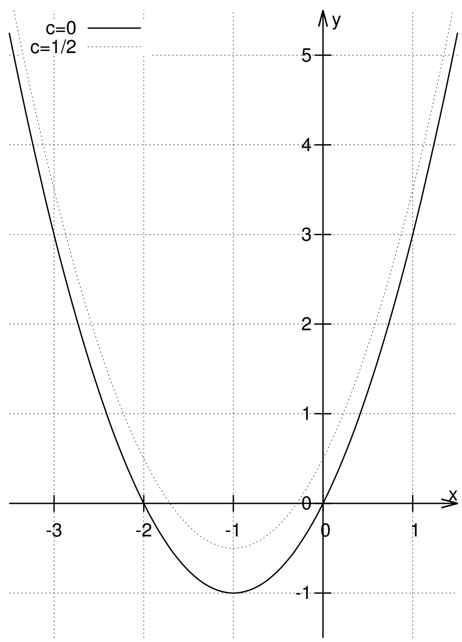
$$2x\frac{dx}{dc} + 2\frac{dx}{dc} + 1 = 0.$$

Solving for the derivative gives

$$\frac{dx}{dc} = -\frac{1}{2x + 2}.$$

At $x = 0$, we have $dx/dc = -1/2$ so this root decreases as c increases. At $x = -2$, we have $dx/dc = 1/2$ so this root increases.

c) As c increases, the parabola is shifted up and the roots move towards $x = -1$.



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1.5 Exercises

1. Find dy/dx when x and y are related as follows:

   (a) $y^2 + xy = 2$

   (b) $e^{xy} + xy + x^3 = xy^2$

   (c) $\sin(xy) + cos(xy) = 1/2$

2. Find dx/dz when x and z are related as follows.

   (a) $x^2 - z^2 = 1$

   (b) $x^2 + axz + x\sin(z) = 2$

3. Consider the curve defined by $x^2 + xy = 3$.

   (a) Find the value(s) of x when $y = 2$.

   (b) Find all tangent lines to the curve at points (x, y) with $y = 2$.

4. Let $y = \tan^{-1}(x)$ be the inverse tangent or arctangent function. Find the derivative dy/dx by applying implicit differentiate to the equation

$$x = \tan(y).$$

   This provides another way to understand our method for finding derivatives of inverse functions.

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