# 1 Lecture 24: Linearization

## 1.1 Outline

- The linearization of a function at a point $a$.

- Linear approximation of the change in $f$. Error, absolute error.

- Examples

## 1.2 Linearization

Functions can be complicated. Often, it is useful to replace a function by a simpler function. Today we will discuss one way to approximate a function and look at how to use this linearization to approximate functions and also when this is a bad idea.

Given a differentiable function $f$ defined near $a$, the *linearization of $f$ at $a$* is the linear function given by

$$L(x) = f(a) + f'(a)(x - a).$$

Thus, the graph of this function is the tangent line to the graph of $f$. We expect that the linearization will be a good approximation to $f$ near $a$, but not a good approximation when we are far away from $a$.

*Example.* Let $f(x) = \sin(x)$. Find the linearization of $f$ at $x = 0$. Use the linearization to approximate $f(0.1)$ and $f(100)$. Compare these approximations with the approximations from your calculator.

*Solution.* To find the linearization at 0, we need to find $f(0)$ and $f'(0)$. If $f(x) = \sin(x)$, then $f(0) = \sin(0) = 0$ and $f'(x) = \cos(x)$ so $f'(0) = \cos(0)$. Thus the linearization is

$$L(x) = 0 + 1 \cdot x = x.$$

Using this linearization, we obtain

| $x$ | $\sin(x)$ | $L(x)$ | $L(x) - \sin(x)$ |
|---|---|---|---|
| 0.001 | 0.0009999998 | 0.001 | $1.67 \times 10^{-10}$ |
| 0.01 | 0.00999983 | 0.01 | $1.67 \times 10^{-7}$ |
| 0.05 | 0.04998 | 0.01 | $2.08 \times 10^{-5}$ |
| 0.1 | 0.099384 | 0.1 | $1.67 \times 10^{-4}$ |
| 0.2 | 0.19867 | 0.2 | $1.33 \times 10^{-3}$ |
| 100 | -0.50637 | 100 | 100.51 |

As expected the linearization is pretty good near 0. It is interesting to note that the error decreases very rapidly as we approach 0. It appears that the error in approximating $\sin(x)$ is approximately $x^3$. Why is this? ∎

We give a few other uses of the linear approximation.

1. In studying the behavior of a pendulum, the angular displacement $\theta$ as a function of time satisfies the equation

$$\theta'' = -k^2 \sin(\theta).$$

   This is a difficult equation to solve and it is common to replace the term $\sin(\theta)$ on the right by the simpler term $\theta$, the linearization of $\sin(\theta)$. This gives the equation

$$\theta'' = -k^2 \theta$$

   and it is easy to see that $\sin(kt)$ and $\cos(kt)$ are solutions. Observations suggest that these functions have the right behavior and that the approximation is useful, when $\theta$ is small.

2. In the examples below, we will use linearization to give an easy way to compute approximate values of functions that cannot be computed by hand. Next semester, we will look at ways of using higher degree polynomials to approximate a function.

3. In a week or two, we will use a linear approximation to help solve an equation such as $f(x) = 0$.

## 1.3   Approximating the change in a function.

Since the linear approximation is only good near a point $a$, it often makes more sense to to talk about the change in $f$. This if we are given a number $a$ and a nearby number $x$, we denote the *change in $f$* by

$$\Delta f = f(x) - f(a).$$

If we replace $f$ by its linearization, $L$, then we obtain the *linear approximation to the change in $f$*,

$$f(x) - f(a) \approx L(x) - L(a) = f'(a)(x - a).$$

It is also common to use the notation $\Delta x = x - a$ for the change in $x$ and the notation $\Delta f = f(x) - f(a)$ for the change in $f$.

We can use the linearization to estimate values functions such as $\sqrt{9.1}$. The point is that we are near a value 9 where it is easy to compute the square root. While this is not terribly useful when we have a calculator, it is good practice and good fun.

We will also compute the error and the percentage error. If a number $a$ is an approximation to the number $A$, then the *error in approximating $A$* is the absolute value of the difference,

$$|a - A|.$$

The percentage error in the approximation of $A$ by $a$ is the quantity

$$100 \times \frac{|a - A|}{|A|}.$$

Often the percentage error is more interesting. If the right answer is 9,000,000, it is probably ok to have an error of 0.1. If the right answer is 0.0001, it is not so helpful if we commit an error of 0.1. [1]

*Example.* Let $f(x) = \sqrt{1 + 2x}$ and use the linearization to approximate $f(4.3)$.
Find the error in the approximation of $f(4.3)$, the percentage error in the approximation of $f(4.3)$ and the percentage error in the approximation of $f(4.3)$.

*Solution.* Note that it is easy to compute $f(4) = \sqrt{9} = 3$. We compute the linearization of $f$ at 4. To do this we need $f(4)$ and $f'(4)$. We have $f(4) = 9$ and we need to find $f'(4)$.
We compute the derivative by the power rule and chain rule,

$$\begin{aligned} \frac{d}{dx}\sqrt{1 + 2x} &= \frac{d}{dx}(1 + 2x)^{1/2} \\ &= \frac{1}{2}(1 + 2x)^{-1/2}2 \\ &= \frac{1}{\sqrt{1 + 2x}}. \end{aligned}$$

Evaluating at $x = 4$, give $f'(4) = 1/3$. Thus the linearization of $f$ at 4 is

$$L(x) = f(4) + f'(4)(x - 4) = 3 + \frac{1}{3}(x - 4).$$

To make the approximation we write

$$f(4.3) \approx L(4.3) = f(4) + \frac{1}{3}0.3 = 3.1.$$

A calculator gives the more accurate approximation,

$$f(4.3) \approx 3.0984.$$

The errors are

$$|f(4.3) - 3.1| \approx 0.00161, \qquad 100\% \times \frac{|f(4.3) - 3.1|}{|f(4.3)|} \approx 0.0521\%.$$

Finally, the percentage error in $f(4.3)$ is given by

$$100\% \times \frac{|f(4.3) - f(4) - f'(4)(4.3 - 4)|}{|f(4.3)|} \approx 0.052\%.$$

∎

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[1] Be careful on the homework. When computing the percentage error, be careful to use the right denominator. Read the question.

*Exercise.* Find the linearization of the function $f(x) = e^{2x}$ at $x = 0$.

*Example.* Suppose that we paint a sphere of diameter 1 meter with a layer of point that is 0.2 mm thick. Estimate the amount of paint used. Use the linear approximation and give your answer in cubic centimeters.

Give a geometric explanation for your answer.

*Solution.* First note that since the diameter is one meter, the radius is 0.5 meter and if we convert 0.2 mm to meters, we have 0.2 mm $= 0.2 \times 10^{-3}$ m $= 0.0002$ m. If $V(r)$ is the volume of a sphere of radius, then the answer to the question is

$$V(0.5002) - V(0.5).$$

We recall that the volume of a sphere is $V(r) = 43\pi r^3$, then the linearization at $r = a$ is

$$L(r) = \frac{4}{3}\pi a^3 + 4\pi a^2(r - a).$$

Thus we have

$$V(0.5002) - V(0.5) \approx 4\,\pi\,0.5^2\,0.0002 \approx 6.28 \times 10^{-4}\mathrm{m}^3.$$

Converting this to cubic centimeters gives about $62.8\mathrm{cm}^3$ of paint.

There is a nice, simple interpretation of this answer. If you cover a surface with a thin layer of paint, the volume of the layer is approximately the area of the surface times the thickness of the layer. For a flat surface, this gives the exact answer. For a curved surface, such as a sphere, this is only an approximation. The error tells us something about the curvature of the surface. But that is a topic for another course.

∎

We close with another reason to consider linear approximation. For a curve which is given implicitly as the solution of an equation such as $x^2 + y^3 = xy$, it is difficult to find points on the on the curve. If one can find one point on the curve, we can use linear approximation to approximate nearby points.

*Example.* Suppose that a curve is given by the equation $x^2 + y^3 = 2x^2y$. Verify that the point $(x, y) = (1, 1)$ lies on the curve. Assume that the curve is given by a function $y = y(x)$ for $x$ near 1 and approximate $y(1.2)$.

*Solution.* To verify that $(x, y) = (1, 1)$ lies on the curve, we need to know that $1^3 + 1^2 = 2 \cdot 1^2 \cdot 1$ which is true.

To find the linearization, we use that $y(1) = 1$ and find the derivative of $y$ at $x = 1$. Differentiating

$$(x^2 + y^3)' = (2x^2y)'$$

gives

$$2x + 3y^2 y' = 4y + 2x^2 y'.$$

Solving for $y'$ gives

$$y' = \frac{4y - 2x}{3y^2 - 2x^2}$$

and that $y'(1) = 2$. Thus the linearization of $y$ is $L(x) = 1 + 2(x-1)$ and $L(1.2) \approx 1.4$. Thus the point $(1, 1.2)$ should be close to the curve.

If we substitute this point into the equation $x^2 + y^3 = 2x^2 y$, we find that $1.2^2 + 1.4^3 = 4.184$ and $2 \cdot 1.2^2 \cdot 1.4 = 4.0320$. As these values are close, we expect that $(1.2, 1.4)$ is close to a point on the curve. ∎

We conclude with the graph of a function where the linear approximation is not a good approximation.
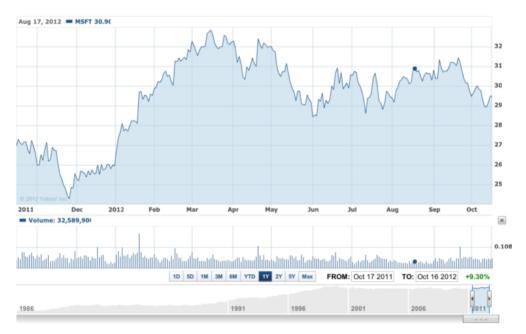


Figure 1: Stock price of MSFT (Microsoft)

## 1.4  Exercises

1. Find the linear approximations of the following functions at the given points.

   (a) $f(x) = x^3 + 2x - 1$ at $x = 1$

   (b) $f(x) = e^{3x-6}$ at $x = 2$

   (c) $f(x) = x \cos(x)$ at $x = 0$

2. Choose an appropriate linear approximation and use it to approximate the following values. Check your answer by computing the exact value on a calculator.

   (a) $e^{0.2}$

   (b) $\sqrt{99}$

   (c) $\sin(45°)$

3. Consider the curve defined by the equation $x^2 + xy + y^3 = 7$. Find the linear approximation to this curve at $(x, y) = (2, 1)$.

4. (a) Suppose $f(x) = x^5 + 4x - 1$, find the linear approximation at $x = 1$.

   (b) If $g$ is the inverse function to $f$, find the linear approximation to $g$ at $x = 2$.

   (c) Is there a relation between the two linear approximations you found in parts a) and b).