1 Lecture 25: Extreme values

1.1 Outline

- Absolute maximum and minimum. Existence on closed, bounded intervals.
- Local extrema, critical points, Fermat's theorem
- Extreme values on a closed interval
- Rolle's theorem

The material in this section has two roles. 1. We will begin studying optimization. How do we find the largest or smallest values of a function? Optimization is an important problem in applied mathematics and engineering: how can we build something is as strong or as cheap as possible? 2. This material is also very important for the next few sections where we establish the relationship between a function and its derivative.

1.2 Absolute maximum and minimum on an interval.

Definition. If f is a function on an interval I, we say that $f(a)$ is an *absolute maximum of f on I* if $f(x) \leq f(a)$ for all x in I.

Exercise. Define what it means for $f(a)$ to be an absolute minimum value for f on I.

The following is an important property of continuous functions.

Theorem 1 *If f is a continuous function on a closed and bounded interval $I = [a, b]$, then f has an absolute maximum value and absolute minimum value on I.*

Note that this is like the intermediate value theorem. It asserts that something happens, but provides little or no help on finding the absolute maximum or minimum value. We will not prove this theorem. The proof is difficult.

Think about the following three examples which show that the theorem may fail if f is not continuous, or if the interval is not closed and bounded. We consider three functions:

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases},$$

The function f is not continuous and has an absolute minimum value, but no absolute maximum value.

$$g(x) = 1/x, \qquad 1 \leq x < \infty,$$

The function g is continuous, defined on an unbounded interval, and has an absolute maximum value, but no absolute minimum value.

$$h(x) = x, \qquad 0 < x \leq 1.$$

The function h is continuous and defined on an open interval. It has neither an absolute maximum value nor an absolute minimum value.

Exercise. Can you find a function defined on the real line which has neither a maximum or minimum value?

Exercise. Can you find a function defined on the real line for which the maximum value is attained at infinitely many points?

1.3 Fermat's theorem, hunting for extreme values

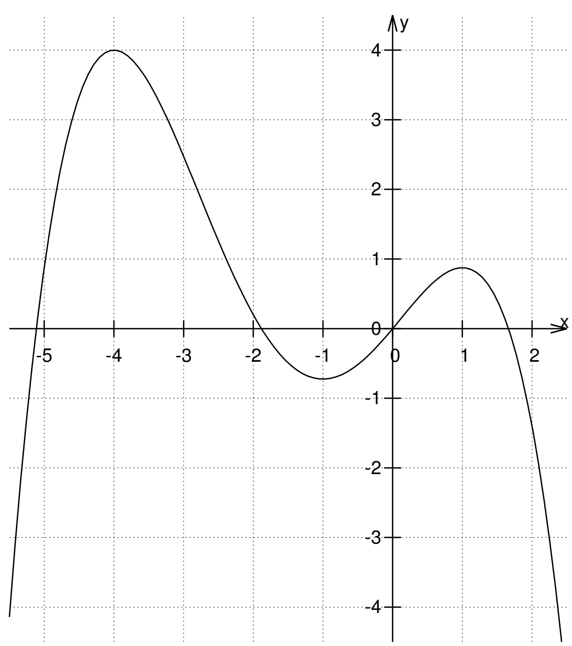
We begin with another definition.

Definition. If f is a function, we say that f *has a local maximum at* a if there is an open interval I in the domain of f so that $f(a)$ is an absolute maximum value for f on I.

Exercise. Define the phrase "f has a local minimum at a".

We give some examples of local maxima and local minima. We will use the word extrema to mean minima or maxima.

Example. Consider the function f whose graph is below. Give the location of the local extrema and absolute extrema.



Solution. It appears that $f(-4)$ is an absolute maximum value for f. In addition f has a local maximum at $x = -4$ and $x = 1$ and a local minimum at $x = -1$. The function does not have an absolute minimum value. ∎

From the interpretation of the derivative as the slope of the tangent line, we expect that at a local maximum or minimum, we have that the derivative is zero. There is one thing that can go wrong. The function may not be differentiable.

Example. If $f(x) = |x|$, then f has a local minimum at 0, but the derivative does not exist there.

Theorem 2 *If f has a local extremum at b, then either $f'(b) = 0$ or $f'(b)$ does not exist.*

Proof. Suppose that b is local maximum and consider the difference quotient at b. We know there is an interval (a, c) with $a < b < c$ so that $f(x) - f(b) \leq 0$ if $a < x < c$. Thus,

$$\frac{f(x) - f(b)}{x - b} \geq 0, \qquad x < b, \qquad \frac{f(x) - f(b)}{x - b} \leq 0, \qquad x > b.$$

If the one-sided limits of the difference exist, then we have

$$\lim_{x \to b^-} \frac{f(x) - f(b)}{x - b} \geq 0, \qquad \lim_{x \to b^+} \frac{f(x) - f(b)}{x - b} \leq 0$$

Thus, if the derivative exists, the limits must give the same value and the value must be zero. ∎

This gives us a place to hunt for extreme values of functions.

Definition. A number c in the domain of a function is a critical point if either $f'(c) = 0$ or f is not differentiable at c.

Thus Fermat's theorem tells us that if f has a local extremum at c, then c is a critical number. It is important to remember that the converse is false. There are critical numbers where f does not have a local extremum.

Example. Find the critical numbers for the following functions. Use a graph to determine if the function has a local extreme at each critical number.

$$f(x) = x^3, \qquad g(x) = -x^2, \qquad h(x) = |x|, \qquad k(x) = \sqrt[3]{x}.$$

Solution. Each function has one critical number at 0. For f and g, the derivative is zero. For h and k, the derivative does not exist at 0.

The function f does not have local maximum or minimum at 0.
The function g has a local (and absolute maximum) at 0.
The function h has a local minimum at 0.
The function k does not have a maximum or minimum at 0.

∎

1.4 Extreme values on closed intervals

Our first theorem today, states that any continuous function on a closed interval has a maximum value and a minimum value. To find the extreme values, we can use the following procedure to find the extreme values of f on an interval $[a, b]$.

1. Find the critical numbers in (a, b) and list endpoints, a and b.

2. Evaluate the function at each of the numbers listed in step 1.

3. Choose the largest and smallest values. These must be the endpoints.

To understand why this procedure works, we note our Theorem 1 tells us that there must be a maximum value and minimum value. If these extremes occur in (a, b), then they are also local minimum values and Fermat's theorem tells us they are critical points. Thus all of the extremes must occur at either an endpoint, or a critical point.

Example. Let $f(x) = xe^{-x}$. Find the absolute maximum and minimum for this function on the interval $[0, 2]$.

Solution. The function f is continuous and differentiable and thus we know there will be an absolute maximum and minimum value on the closed interval $[0, 2]$. We use the product to compute the derivative $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$. The only critical number is $x = 1$. We make a table of the values of f at the endpoints and the critical point,

x	$f(x)$
0	0
1	$1/e \approx 0.37$
2	$2/e^2 \approx 0.27$

Thus the absolute minimum value of f is $f(x) = 0$ and the absolute maximum value for f is $f(1) = 1/e$. ∎

Exercise. Find the absolute maximum and minimum values of $f(x) = |x-1| - \frac{1}{2}|x-2|$ for f in the interval $[0, 3]$.

1.5 Rolle's theorem

We conclude with an important result that will be needed in the next few days.

Theorem 3 *If f is a continuous function on a closed interval $[a, b]$, f is differentiable on (a, b), and $f(a) = f(b)$, then there is a number c in (a, b) where $f'(c) = 0$.*

Proof. To see why this is true, Theorem 1 tells us that f must have an absolute maximum value and absolute minimum value in the closed interval $[a, b]$. If one of these occurs at a point c in (a, b), then f has a local maximum or minimum at c and Fermat's theorem, Theorem 2, tells that $f'(c) = 0$.

If both the absolute maximum and minimum value occur at the endpoints, then our assumption $f(a) = f(b)$ tells us that the function is constant and the derivative is zero in all of (a, b). ∎

1.6 Exercises

1. Determine if the following functions have an absolute maximum on the given domain. If a maximum value exists, find it.

   (a) $\sin(x)$ on the interval $[\pi/4, \pi]$.

   (b) $\sin(x)$ on the interval $(0, \pi)$.

   (c) $x^2 - 3x$ on the interval $[-1, 5]$.

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