1 Lecture 33: Area

- Area and distance traveled
- Approximating area by rectangles
- Summation
- The area under a parabola

1.1 Area and distance

Suppose we have the following information about the velocity of a particle, how can we estimate the distance travelled?

Time (seconds)	0	3	5	6
Velocity (meters/second)	2	4	3	5

With this little information, it is impossible to know the answer. One possibility is to assume that the velocity in each interval is the same as at the left endpoint. The table below summarizes the calculation

Interval	Length of interval	Velocity	Distance
$[0, 3]$	3 s	2 m/s	6 m
$[3, 5]$	2 s	4 m/s	8 m
$[5, 6]$	1 s	3 m/s	3 m
Total			17 m

The picture in figure 1.1 helps us to see that the area represents the distance traveled

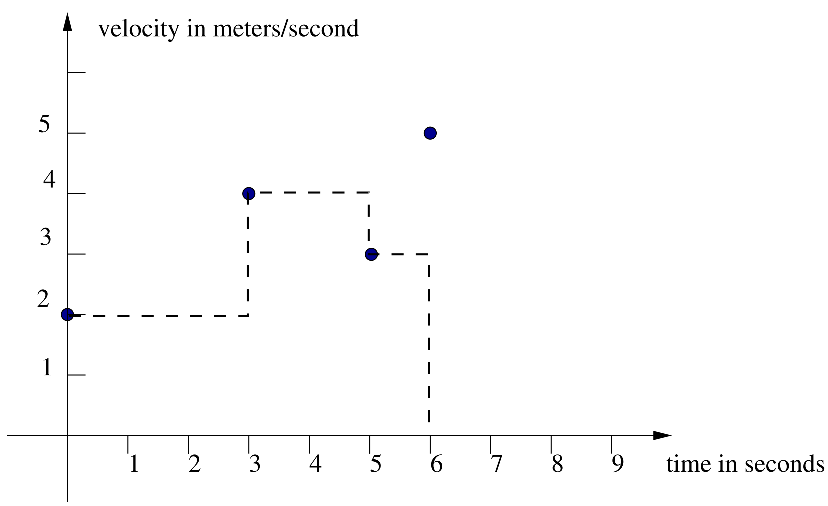

Figure 1: Area represents distance travelled

Can you think of a different way to estimate the distance? A better way?

1.2 Approximating area by rectangles

In this section, we will consider a way of approximating the area below the graph of a function. For example, we might be interested in finding the area of the region $R = \{(x, y) : 1 \leq x \leq 3, 0 \leq y \leq x^2\}$ which lies under the graph of a parabola.

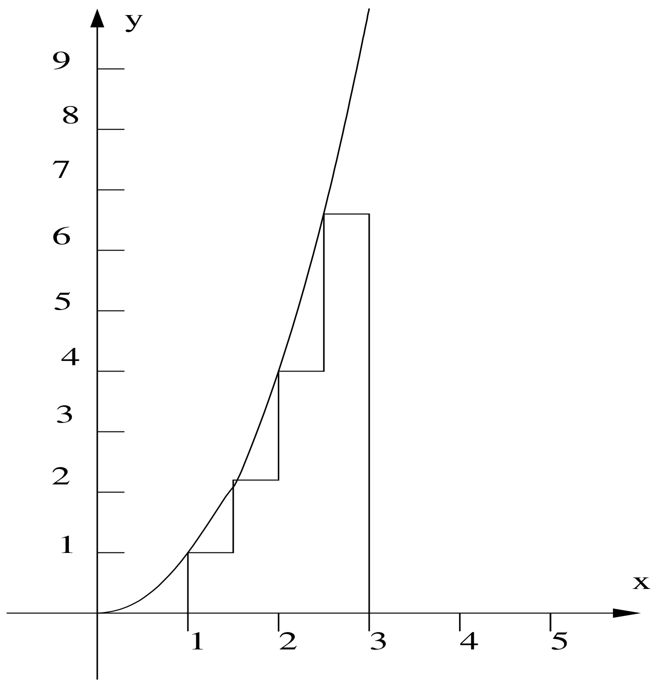

Figure 2: The left endpoint approximation to the area under the parabola $y = x^2$ for $1 \leq x \leq 3$

To do this, we will first divide the region into several intervals of equal length. If we decide to use four intervals, we will use the division points $x_0 = 1$, $x_1 = 3/2$, $x_2 = 2$, $x_3 = 5/2$, and $x_4 = 3$. For each interval, we choose a point in the interval and use the value of f at that point to give the height of a rectangle. Finally, we add the areas of the rectangles to give the value of f. It is not so important which value we choose. Eventually, we will take a limit as the intervals become small and the result of different choices will be small.

For now, there are three common choices: for each interval choose the left endpoint, the right endpoint or the midpoint. These three choices give us three different approximate values for the area:

$$
\begin{aligned}
R_4 &= \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2) \\
&= \frac{1}{2}(\frac{9}{4} + 4 + \frac{25}{4} + 9) = \frac{43}{4} = 10.75
\end{aligned}
$$

$$\begin{aligned} L_4 &= \frac{1}{2}(1+\frac{9}{4}+4+\frac{25}{4}) = \frac{27}{4} = 6.75 \\ M_4 &= \frac{1}{2}((5/4)^2+(7/4)^2+(9/4)^2+(11/4)^2) = 69/8 = 8.625. \end{aligned}$$

We are using R_n to denote the sum formed using the right endpoint and n intervals, L_n if we use the left endpoint to fix the height of the rectangles and M_n if we use the mid-point.

Figure 2 illustrates the computation of L_4.

1.3 Summation

Computing areas requires that we have an efficient way to evaluate sums. If a_k gives a number for integer values of k, then we use the notation

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \ldots + a_n.$$

For example,

$$\sum_{k=1}^{5} k^2 = 1+4+9+16+25 = 55.$$

Example. Express as a sum

$$1+4+7+10+13+16.$$

Solution. Since the terms differ by 3, we may use a linear expression to generate these numbers. If we want to start at $k=1$ and have the first term 1 the linear expression we need is $3(k-1)+1 = 3k-2$. This is similar to using the point-slope form to find the equation of a line. We need to find n so that $3n-2=16$ or $n=6$. Thus, we have

$$1+4+7+10+13+16 = \sum_{k=1}^{6}(3k-1).$$

∎

Example. Find the sum

$$\sum_{k=1}^{7}(2k-1).$$

Solution. The expression $2k-1$ will give us the odd numbers. Thus we have

$$\sum_{k=1}^{7}(2k-1) = 1+3+5+7+9+11+13 = 49.$$

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There are several formulae for sums that are quite useful. We give them without proof

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{1}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{2}$$

$$\sum_{k=1}^{n} k^3 = \frac{(n(n+1))^2}{4} \tag{3}$$

Example. Find the value of the sum

$$\sum_{k=1}^{20}(3k+2).$$

Solution. Using the distributive law, we have

$$\sum_{k=1}^{20}(3k+2) = 3\sum_{k=1}^{20} k + \sum_{k=1}^{20} 2.$$

Since $\sum_{k=1}^{20} 2$ tells us to add together 20 copies of the number 2, its value is 40. According to (1), the value of the sum $\sum_{k=1}^{20} k$ is $\frac{20\cdot 21}{2} = 210$. Thus we have

$$3\sum_{k=1}^{20} k + \sum_{k=1}^{20} 2 = 3 \cdot 210 + 40 = 670.$$

∎

1.4 A definition of area

If f is a positive function and $[a, b]$ is a closed interval, our goal is to define the area of $R = \{(x, y) : a \le x \le b, 0 \le y \le f(x)\}$.

We take an interval $[a, b]$ and divide it into n equal subinterval. The length of each subinterval is $(b - a)/n$. The division points will be

$$x_0 = a, x_1 = a+\frac{b-a}{n}, x_2 = a+2\frac{b-a}{n}, \ldots, x_{n-1} = a+(n-1)\frac{b-a}{n}, x_n = a+(b-a) = b$$

or the kth division point is $x_k = a + \frac{k}{n}(b - a)$.

Using these points, we can write out the right endpoint, left endpoint, and midpoint approximations to area.

$$\begin{aligned}
R_n &= \frac{b-a}{n} \sum_{k=1}^{n} f(x_k) \\
L_n &= \frac{b-a}{n} \sum_{k=0}^{n-1} f(x_k) \\
M_n &= \frac{b-a}{n} \sum_{k=1}^{n} f(\frac{1}{2}(x_{k-1}+x_k)).
\end{aligned}$$

Finally, if we let $n \to \infty$, we expect that the error will be quite small and disappear in the limit. Thus if R is the region $\{(x,y) : a \leq x \leq b, 0 \leq y \leq f(x)\}$, then any of the limits

$$\lim_{n\to\infty} R_n, \qquad \lim_{n\to\infty} L_n, \qquad \lim_{n\to\infty} M_n$$

is a good candidate for the area of R. What we would like is for each of these to give the same answer.

Example. Use the right and left endpoint approximations with 10 subdivisions to estimate the area bounded by the graph of $y = x^3$, $y = 0$, and $x = 2$.

Use a sketch to determine if each estimate is larger or smaller than the desired area.

Solution. We begin by sketching the region and see that we need to subdivide the interval $[0, 2]$ into 10 equal sub-intervals. Each will be of length $1/5$. The division points will be $x_k = k/5$ for $k = 0, 1 \ldots, 10$.

The left and right sums are

$$R_{10} = \frac{1}{5} \sum_{k=1}^{10} \left(\frac{k}{5}\right)^3 \text{ and } L_{10} = \frac{1}{5} \sum_{k=0}^{9} \left(\frac{k}{5}\right)^3.$$

We simplify the right sum and obtain that

$$R_{10} = \frac{1}{5^4} \sum_{k=1}^{10} k^3 = \frac{(10 \cdot 11)^2}{4 \cdot 625} = 4.84.$$

Simplifying the left sum, we obtain

$$L_{10} = \frac{1}{5^4} \sum_{k=0}^{9} k^3 == \frac{1}{5^4} \sum_{k=1}^{9} k^3 \frac{(9 \cdot 10)^2}{4 \cdot 625} = 3.24.$$

From a sketch we see that the right sum gives the area that is larger than the desired area and the left sum gives a smaller area. ∎

Example. Find the area under the graph of $f(x) = x^2$ for $0 \leq x \leq 3$. The area of the region $\{(x, y) : 0 \leq y \leq 3, 0 \leq y \leq x^2\}$ using the right endpoint approximation.

Solution. We use the division points $x_k = k/n$. The length of each interval is $3/n$. Thus the the right endpoint approximation to the area is

$$R_n = \frac{3}{n} \sum_{k=1}^{n} (\frac{3k}{n})^2.$$

Using the formula (2) we have that the sum is

$$\begin{aligned} R_n &= \frac{3}{n} \sum_{k=1} \frac{9}{n^2} k^2 \\ &= \frac{27}{n^3} \sum_{k=1} k^2 \\ &= \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

Finally, we may take a limit and find a candidate for the area

$$\lim_{n \to \infty} \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6} = 9.$$

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