10/21/15.

Linear approximation

Given a function f and a point a, find a linear function $L(x)$ which approximates f near a.

Linearization of f at a is

$$L(x) = f(a) + f'(a)(x-a).$$

Find the linearization of $f(x) = \sin(x)$ at $x = 0$.

Need $f(0) = \sin(0) = 0$

$f'(x) = \cos(x) \quad f'(0) = 1$

$L(x) = 0 + 1\cdot(x-0)$

$= x$

Use the linearization to approximate $\sin(0.01)$

$\sin(0.1)$ & $\sin(100)$

x	$\sin(x)$	$L(x) - \sin(x)$
0.001	0.0009999998	1.67×10^{-10}
0.01	0.0099983	1.67×10^{-7}
0.1	0.099384	1.67×10^{-4}
100	$\sin(100) = -0.506$	100.51

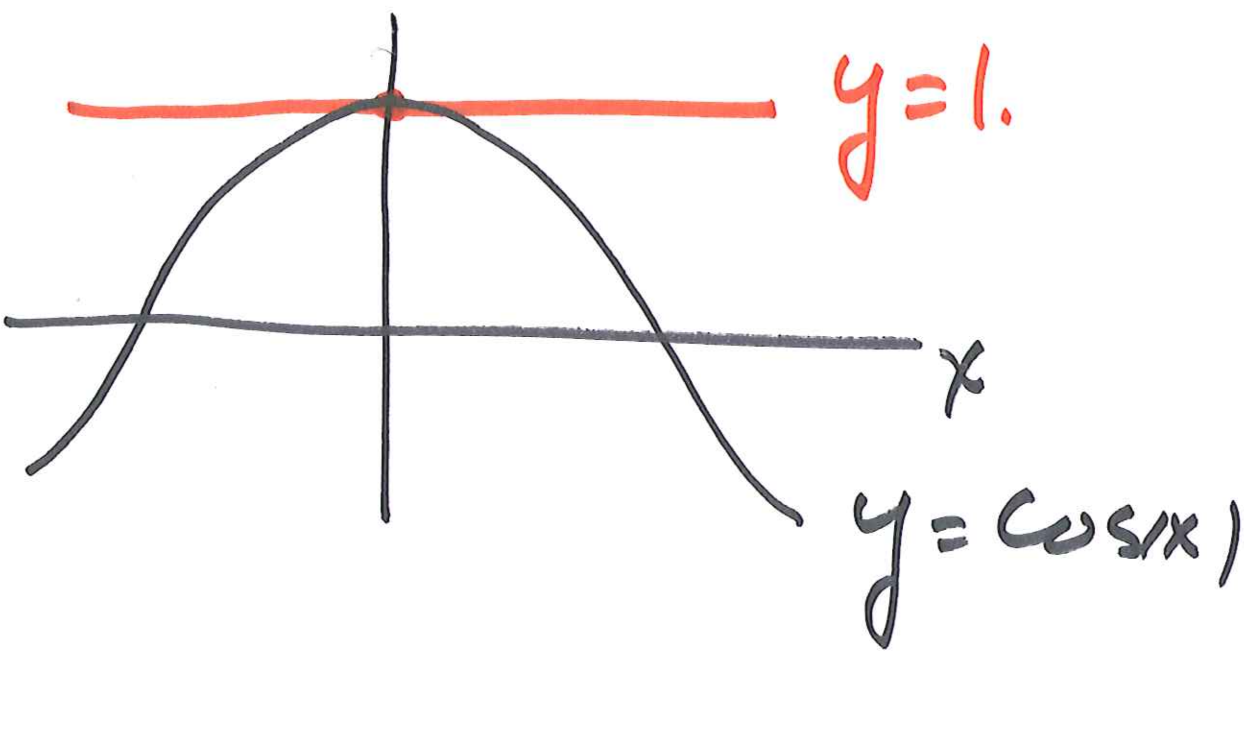
$f(x) = \cos(x)$. Find linearization at 0.

$f'(x) = -\sin(x)$.

$f(0) = 1$, $f'(0) = 0$.

$L(x) = f(0) + f'(0)(x-0)$

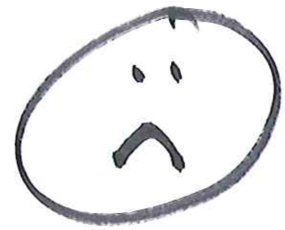
$= 1 + 0 = \underline{1}$



Why?

$$\theta'' = -k^2 \sin(\theta).$$

Pendulum



Hard to solve.

Change the problem to an easier problem.

Replace $\sin\theta$ by its linearization.

$$\theta'' = -k^2 \theta$$

$$\theta(t) = A\cos(t) + B\sin(t).$$

2^{nd} application -
Approximating numerical values of functions.

$$f(x) \approx L(x)$$

$$f(x) \approx f(a) + f'(a)(x-a)$$

or $$f(x) - f(a) \approx f'(a)(x-a).$$

$$\Delta f \approx f'(a)\Delta x.$$

Let $f(x) = \sqrt{1+2x}$.

Use the linearization to approximate $f(4.3)$

Find the error & %-error.

Observe that

$$f(4) = \sqrt{1+8} = 3$$

$$f(4.3) \approx f(4) + 0.3 f'(4).$$

Use the linear approx. at 4.

$$f'(x) = 2 \frac{1}{2}(1+2x)^{-1/2}$$

$$= (1+2x)^{-1/2}.$$

$f'(4) = (1+8)^{-1/2} = \frac{1}{\sqrt{9}} = \frac{1}{3}.$

$f(4.3) \approx \cancel{3}\ 3 + 0.3\ \frac{1}{3}$

$= 3.1$

Error. $|f(4.3) - 3.1|.$

$\approx |3.0984 - 3.1| \approx 0.00161$

Percent error in ~~g~~ ~~an approximat~~ approximating A by a

$\frac{|A-a|}{|A|} \cdot 100\%$

$\frac{|f(4.3) - 3.1|}{|f(4.3)|} \cdot 100\%$

$\approx 0.0521\%$

Example. We have a sphere of diameter 1m. We apply a layer of paint that is 0.2 mm thick. How much paint do we use?

2 answers. Before painting volume $\frac{4}{3}\pi r^3$ w/ $r = \frac{1}{2}$ meter $\frac{4}{3}\pi \frac{1}{8} = \frac{\pi}{6} m^3$

After painting radius is

$\frac{1}{2} + \frac{0.2}{1000}$ m $= 0.5002$.

$V(r) = \frac{4}{3}\pi r^3 \quad V'(r) = 4\pi r^2$

$V(0.5002) - V(0.5)$

$\approx 6.28 \cdot 10^{-4}\ m^3$

Linearization.

$V(0.5002) - V(0.5)$

$\approx V'(0.5) \cdot 0.0002$

$\approx 6.28 \cdot 10^{-4}\ m^3$

Consider

$$x^2 + y^3 = 2x^2 y$$

- Verify $(x,y) = (1,1)$ lies on this curve
- Find the linearization of $y = y(x)$ near $(x,y) = (1,1)$

Implicit differentiation.

$$2x + 3y^2 y' = 4xy + 2x^2 y'$$

Substitute $(x,y) = (1,1)$.

$$2 + 3y' = 4 + 2y'$$

$$y' = 2.$$

Linearization is

$y - 1 = 2(x - 1)$.

Approximate

$\sqrt{27} - \sqrt{25}$

$f(x) = \sqrt{x}$.

$f'(x) = \frac{1}{2\sqrt{x}}$

Clicker #3 (corrected)

Approximate $\sqrt{27} - \sqrt{25}$.

Let $f(x) = \sqrt{x}$. We want to compute $f(27) - f(25)$. If we use the linear approximation at 5, we obtain

$$f(27) - f(25) = f'(25) \cdot 2$$

$$= \frac{1}{10} \cdot 2 = \underline{\underline{0.2}}$$