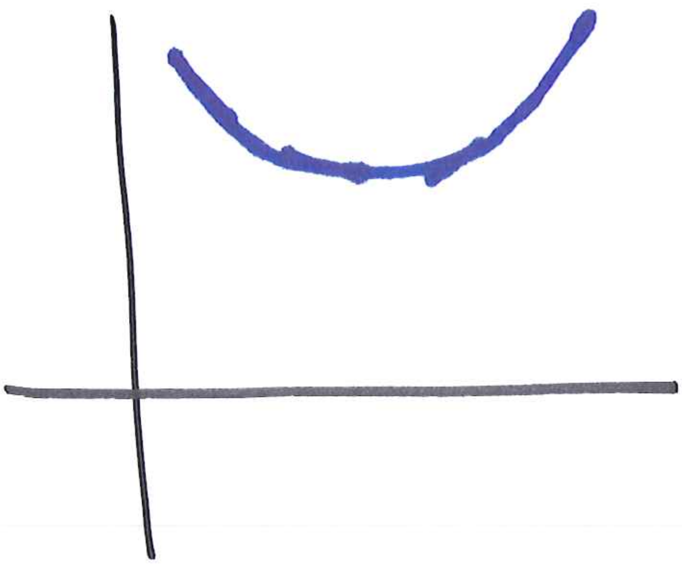
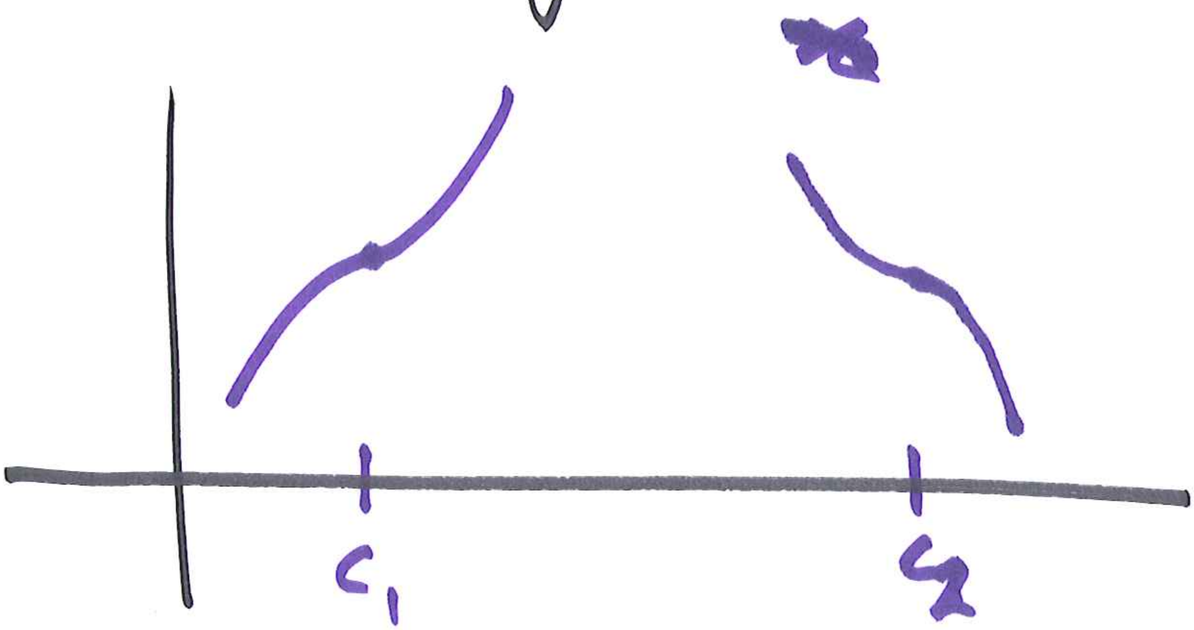
L27. Shape of a graph.

<u>Def</u>. ~~A f~~ The graph of a function is <u>concave</u> <u>up</u> on an interval I if $f'$ is increasing on I.



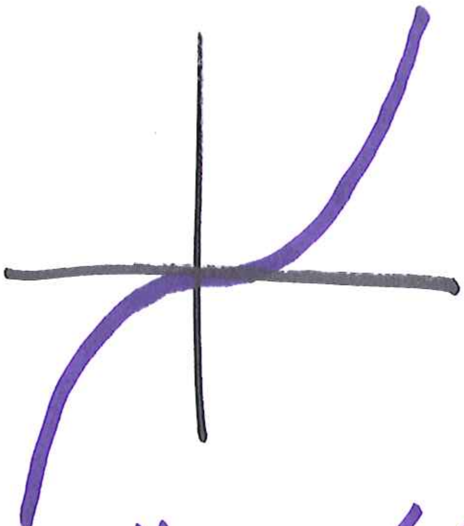
<u>Exercise</u>. Define <u>concave</u> <u>down</u>

__Definition__ A point $(c, f(c))$ on the __graph__ of $f$ is called an __inflection point__ if the graph changes concavity at $c$



# Examples

$f(x) = x^3$

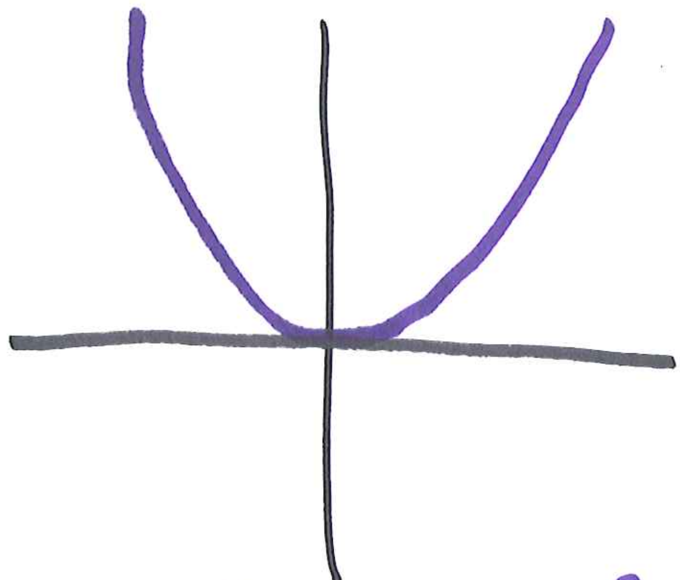


$f''(x) = 6x$

$f''(0) = 0$ and

$(0, f(0))$ is an inflection point

$f(x) = x^4$



$f''(x) = 12x^2$

$f''(0) = 0$

$(0, f(0))$ is not an inflection point.

Test for concavity.

Suppose $f$ is a function and $f''$ exists on the interval $(a, b)$.

- If $f''(x) > 0$ on $(a,b)$, then $f$ is concave up on $(a, b)$
- If $f''(x) < 0$ on $(a, b)$, then $f$ is concave down.
- If $c$ is in the interval and $f''$ changes signs at $c$, then $(c, f(c))$ is an inflection point.

$f(x) = x^4$

$f'(x) = 4x^3$

$f''(x) = 12x^2$

$f''(0) = 0$, but $12x^2 \geq 0$ so

0 is not an inflection point.

Example. Suppose $f$ is a function and $f'(x) = x^3 - 3x^2$

- Find intervals of increase & decrease
- Classify critical points as local extrema.
- Find intervals of concavity.

$$f'(x) = x^3 - 3x^2 = x^2(x-3).$$

$$f'(0) = 0 \text{ and } f'(3) = 0$$

Test sign

| | $(-\infty, 0)$ | $(0, 3)$ | $(3, \infty)$ |
|---|---|---|---|
| $x^2$ | + | + | + |
| $x-3$ | − | − | + |
| $x^2(x-3)$ | − | − | + |
| | ↘ | ↘ | ↗ |

f is decreasing on

$(-\infty, 0)$ and $(0, 3)$.

increasing on $(3, \infty)$.

crit. pts. are $x = 0, 3$

f has a local min. at 3

0 is neither max. nor min

$$f''(x) = 3x^2 - 6x$$

$$= 3x(x-2).$$

$f''(x) = 0$ if $x = 0$ or $2$

| | $(-\infty, 0)$ | $(0, 2)$ | $(2, \infty)$ |
|---|---|---|---|
| $3x$ | $-$ | $+$ | $+$ |
| $x-2$ | $-$ | $-$ | $+$ |
| $f''$ | $+$ | $-$ | $+$ |
| | $\cup$ | $\cap$ | $\cup$ |

f is concave up on

$(-\infty, 0)$ and $(2, \infty)$

concave down on $(0,2)$

Inflection points - at $(0, f(0))$ and $(2, f(2))$.

Example (Cliche).

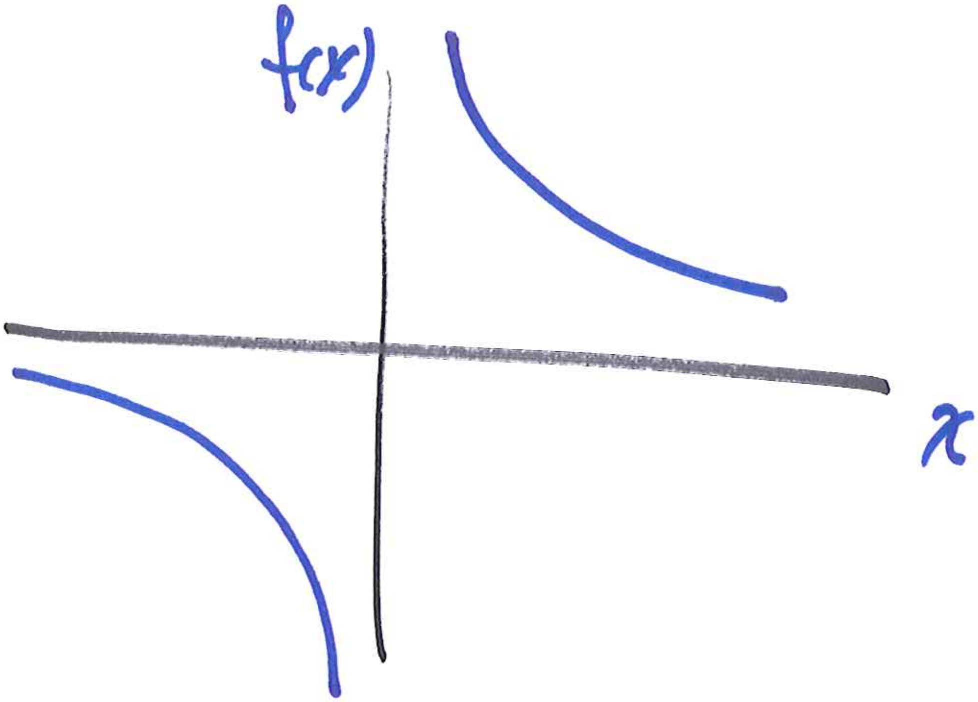
If $f(x) = 1/x$, find concavity

$f'(x) = -1/x^2$, $f''(x) = 2/x^3$.

$f''(x) > 0$ on $(0, \infty)$ $\cup$

$f''(x) < 0$ on $(-\infty, 0)$ $\cap$

0 is not in the domain, so $(0, f(0))$ is not an inflection point.



2nd deriv. test for local extrema.

Suppose $f'(c) = 0$ and $f''(x) > 0$ near $c$. Then $f'$ is increasing. Since it goes through 0, $f'$ changes from ~~pos~~ negative to positive. or $f$ goes from decreasing to increasing. So $c$ is a local min.

~~T~~ Theorem. Suppose $f'(c)=0$ and $f''$ is continuous near $c$, ~~then~~

- If $f''(c)>0$, $f$ has a local min at $c$.
- If $f''(c)<0$, $f$ has a local max at $c$.
- If $f''(c)=0$, ~~th~~ no information.

<u>Example</u> $f'(x) = x^3 - 3x^2$

Use $2^{\text{nd}}$ deriv. to classify crit. pts.

$f'(x) = 0$ at $x = 0, 3$

$$f''(x) = 3x(x-2).$$

$f''(0) = 0$. No information.

$f''(3) = 9 \cdot 1 > 0$ so f has a local min at 3.