# L28. Limits at infinity. — Sect 2.7.

In ch. 2 we talked about $\lim_{x \to a} f(x)$ for a a finite number. Now we let $a = +\infty$ or $-\infty$.

<u>Def.</u> Let $f$ be defined on the interval $(a, \infty)$. We say that

$$\lim_{x \to a} f(x) = L$$

if we can make $f(x) - L$ small by requiring $x$ to be large & positive.

- Define $\lim_{x \to -\infty} f(x) = L$.

- We also allow $L = +\infty$ or $-\infty$.

<u>Example</u>. For which $n$ do we have

$$\lim_{x \to +\infty} x^n = 0$$

$$\lim_{x \to +\infty} x^n = \begin{cases} +\infty, & \text{if } n > 0 \\ 1, & \text{if } n = 0 \\ -\infty & \text{if } n < 0 \end{cases}$$

Example. Find

$\lim_{x \to \infty} x$, $\lim_{x \to -\infty} x^3$

$\lim_{x \to -\infty} \frac{1}{\sqrt{x}}$.

$\lim_{x \to \infty} x = +\infty$, $\lim_{x \to -\infty} x^3 = -\infty$

$\lim_{x \to -\infty} \frac{1}{\sqrt{x}}$ DNE. Since

$\sqrt{x}$ is undefined if $x<0$.

**Example** Let $c$ be your favorite number. Find functions $f$ & $g$ so that

$$\lim_{x \to \infty} f(x) = 0, \quad \lim_{x \to \infty} g(x) = \infty$$

and $\lim_{x \to \infty} f(x) \cdot g(x) = c$.

**Solution** $f(x) = \frac{c}{x}$. $g(x) = x$

$f \cdot g = c$. "$0 \cdot \infty = c$"

**Moral.** Don't multiply $0 \cdot \infty$. (Also, $\infty - \infty$ and $\frac{\infty}{\infty}$ are problems)

These are the only problems.
Otherwise arithmetic
with $\infty$ makes sense.

$a \cdot \infty$, $a \neq 0$ $\quad \infty \cdot \infty = \infty$

$\frac{a}{\infty} = 0$, $a \neq \infty$.

Limit laws hold when
the limit is $\pm\infty$.

<u>Example</u>. Find

$$\lim_{x \to -\infty} \frac{3x+1}{7x+2}.$$

$$\lim_{x \to -\infty} 3x+1 = -\infty$$

$$\lim_{x \to -\infty} 7x+2 = -\infty$$

$$\frac{-\infty}{-\infty} = ?$$

Simplify: $\dfrac{3x+1}{7x+2} = \dfrac{\cancel{x}\left(3+\frac{1}{x}\right)}{\cancel{x}\left(7+{}^{2}/{x}\right)}$

$$\lim_{x \to -\infty} \frac{3x+1}{7x+2} = \lim_{x \to -\infty} \frac{3+\frac{1}{x}}{7+\frac{2}{x}} = \frac{3}{7}.$$

Clicker. $\dfrac{2x+1}{6x-1} = \dfrac{\cancel{2+\frac{1}{x}}}{6(}$

$$= \frac{x\left(2+\frac{1}{x}\right)}{\cancel{x}\left(6-\frac{1}{x}\right)}.$$

$$\lim_{x\to\infty} \frac{2x+1}{6x-1} = \lim_{x\to\infty} \left( \frac{(2+1/x)}{6-1/x} \right)$$

$$= 2/6 = 1/3.$$

<u>Limits</u> of <u>rational</u> functions

$$R(x) = \frac{p_n x^n + \cdots + p_0}{q_m x^m + \cdots + q_0}$$

$$= x^{n-m} \left( \frac{p_n + \cdots + p_0/x^n}{q_m + \cdots + q_0/x^m} \right)$$

As $x \to \infty$

$$\lim_{x \to +\infty} x^{n-m} = \begin{cases} 0 & \text{if } n < m \\ 1 & \text{if } n = m \\ +\infty & \text{if } n > m \end{cases}$$

$$\lim_{x \to +\infty} \frac{p_n + \cdots + p_0/x^n}{q_m + \cdots + q_0/x^m} = \frac{p_n}{q_m}.$$

(We assume $p_n \neq 0$ & $q_m \neq 0$.

Find:

$$\lim_{x \to -\infty} \frac{x^3+1}{2x^2-x} \quad \Big| \quad \lim_{x \to +\infty} \frac{x^2+1}{2x^3-x}.$$

$$\frac{x^3+1}{2x^2-x} = \frac{x^3(1+1/x^3)}{x^2(2-1/x)}$$

$$\lim_{x \to -\infty} x\left(\frac{1+1/x^3}{2-1/x}\right).$$

$$= -\infty \cdot \frac{1}{2} = -\infty.$$

$$\frac{x^2+1}{2x^3-x} = \frac{x^2(1+1/x^2)}{x^3(2-1/x^2)}.$$

$$= \frac{1}{x}\left(\frac{1+\frac{1}{x^2}}{2-1/x^2}\right)$$

$$\lim_{x \to \infty} \left( \frac{1}{x} \left( \frac{1 + \frac{1}{x^2}}{2 - 1/x^2} \right) \right)$$

$$= \bullet \cdot \frac{1}{2} = 0.$$

<u>Example</u> Find

$$\lim_{x \to \infty} \left( x - \frac{x^2 - 2x}{x+1} \right).$$

$$x - \frac{x^2 - 2x}{x+1.} = \frac{x(x+1) - (x^2 - 2x)}{x+1}$$

$$= \frac{x^2 + x - x^2 + 2x}{x+1} = \frac{3x}{x+1.}$$

$$\lim_{x \to \infty} \frac{3x}{x+1} = 3.$$

Clicker.

A. $n=m$, limit is

$$\frac{a_n}{b_m} \neq 0.$$

B. $\frac{a_n}{b_m}$ ~~≠~~ may not be 1.

C. If $m>n$,

$$\lim_{x\to+\infty} \frac{x^n}{x^m} \frac{a_n + \cdots}{b_m + \cdots}$$

$$= 0. \checkmark$$

D. $\lim_{x\to+\infty} \frac{x^n}{x^m} \frac{a_n +}{b_m + \cdots}$ ~~=~~ $-\infty$

if $m<n$, and $\frac{a_n}{b_m} < 0$.