L29. L'Hôpital's rule.

11/2/15.

Limits.

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x-1)}{\cancel{(x-2)}}$$

$$= 1.$$

$$\lim_{x \to 1} \frac{e^x}{x+1} = \frac{1}{2} e,$$ by substitution into a continuous function

$$\lim_{x \to 0} \frac{\ln(1+x) - 0}{x}$$

$$= \lim_{h \to 0} \frac{\ln(1+h) - \ln(1)}{h}$$

$$= f'(1) = 1.$$

$$(\text{or } f(x) = \ln(x),\ f'(x) = 1/x$$

$$\lim_{x \to 0} \frac{\ln(1+x)}{\sin(x)} = 1\ (?).$$

Systematic way to evaluate

$$\lim_{x \to a} \frac{f(x)}{g(x)}, \text{ when}$$

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$

- ~~Ind~~ This is called the indeterminate form $0/0$. Other indeterminate forms: $\frac{\infty}{\infty}$, $0 \cdot \infty$, 1^{∞}.

Example $\lim_{x \to \infty} \frac{e^x}{x}$ is the indeterminate form $\frac{\infty}{\infty}$ since

$$\lim_{x\to\infty} e^x = \lim_{x\to\infty} x = +\infty.$$

<u>Theorem</u> (l'Hôpital).

If f, g are differentiable in an open interval containing a and $f(a) = g(a) = 0$, and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

$$\lim_{x\to 2} \frac{(x^2-3x+2)}{x-2}$$ is

the indeterminate form

$\frac{0}{0}$, so we can use

l'Hopital. $\lim_{x\to 2}(x-2)$

$= \lim_{x\to 2}(x^2-3x+2) = 0.$

$$\lim_{x\to 2} \frac{(x^2-3x+2)}{x-2} = \lim_{x\to 2.} \frac{2x-3}{1}$$

$$= 4-3 = \underline{1.}$$

$\frac{x^2-2x+1}{x^2-1}$ is $\frac{0}{0}$ at $x=1$.

$\frac{1-\cos(x)}{x^2}$ is $0/0$ at 0

D $\frac{x}{e^x}$ $\frac{\infty}{\infty}$ at ∞.

C $\frac{x^2-2x+1}{x^2-1}$ is $\frac{4}{0}$ at $x=-1$.

C is not indeterminate.

Clicker # 3

For which value of A can we use l'Hôpital to evaluate the limit

$$\lim_{x \to 1} \frac{x^2 + x + A}{x - 1}.$$

Need $x^2 + x + A = 0$ at $x = +1$.

$1 + 1 + A = 0$ or $\underline{A = -2}$.

Example

$$\lim_{x \to 0} \frac{\ln(1+x)}{\sin(x)}$$

$\ln(1+0) = 0$, $\sin(0) = 0$.

Thus we have the indeterminate form of $0/0$.

Thus

$$\lim_{x \to 0} \frac{\ln(1+x)}{\sin(x)} = \lim_{x \to 0} \frac{\frac{1}{1+x}}{\cos(x)}$$

$$= 1.$$

$$\lim_{x \to \infty} \frac{x}{e^x}.$$

Is the indeterminate form

$\frac{\infty}{\infty}$. $\quad \frac{d}{dx} x = 1, \frac{d}{dx} e^x = e^x$

So $\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0.$

Find $\lim_{x \to \infty} \frac{x^n}{e^x}$,

for $n = 1, 2, 3, \ldots$

$\cancel{\lim_{x}}$ $\dfrac{x^n}{e^x}$ is the indeterminate form $\dfrac{\infty}{\infty}$, if $n>0$.

Thus

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim \frac{n x^{n-1}}{e^x}$$

$$= \cdots = \lim_{x\to\infty} \frac{n!}{e^x}$$

After n applications of L. H.

$$=0.$$

$$\lim_{x \to 0} \frac{1-\cos(2x)}{x^2}.$$

This is the indeterminate form $\frac{0}{0}$.

$$= \lim_{x \to 0} \frac{2\sin(2x)}{2x}$$

Indet. form $\frac{0}{0}$.

$$= \lim_{x \to 0} \frac{4\cos(2x)}{2}$$

$$= \frac{4}{2} = \underline{\underline{2}}.$$

$\lim_{x \to \infty} \frac{x^n}{e^x}$ if n is not a whole number.

This is the indeterminate form $\frac{\infty}{\infty}$ if $n > 0$. Apply l'Hôpital's rule k times to obtain

$$\lim_{x \to \infty} \frac{x^n}{e^x} = \cdots = \lim_{x \to \infty} \frac{n(n-1)\ldots(n-k)x^{n-k-1}}{e^x}$$

until $n-k-1 < 0$.

Obtain $\lim_{x \to \infty} \frac{x^n}{e^x} = 0$.

L'H + Algebra.

Find $\lim_{x \to 0^+} x \ln(x)$.

$\lim_{x \to 0^+} x = 0$. $\lim_{x \to 0^+} \ln(x) = -\infty$.

$$x \ln(x) = \frac{\ln(x)}{1/x} \quad \left(\frac{\infty}{\infty} \right.$$

indet form

or $\cancel{\frac{x}{1/\ln(x)} \quad \left(\frac{0}{0} \right).}$

1st form, it is easier to do derivatives.

$$\lim_{x \to 0^+} x\ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{1/x}$$

$$= \lim_{x \to 0^+} \frac{1/x}{-1/x^2}$$

by L.H. $= \lim_{x \to 0^+} \frac{-x^2}{x} = -\infty.$

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

Indet. form 1^∞.

$$\left(1 + \frac{1}{x}\right)^x = e^{x \ln\left(1 + \frac{1}{x}\right)}.$$

$$\lim_{x \to \infty} x \ln\left(1 + \frac{1}{x}\right) \qquad (0 \cdot \infty).$$

$$x \ln\left(1 + \frac{1}{x}\right)$$

$$= \frac{\ln\left(1 + \frac{1}{x}\right)}{1/x} \qquad \frac{0}{0} \text{ at } \infty$$

$$\lim_{x \to \infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x} \overset{\lim}{=} \frac{1}{1+\frac{1}{x}} \frac{-1/x^2}{-1/x^2}$$

$$= 1.$$

$$\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x = e^1.$$