Anti-derivatives. 11/9/15 ①

Def. We say F is an <u>anti</u>-<u>derivative</u> of f if

$$F' = f.$$

- Why? Can we recover the position from velocity?

- Do they exist?
  If $f(x) = e^{-x^2}$, can you find F so that $F'(x) = e^{-x^2}$.

- How many?

If $f(x) = \cos(x)$, find an anti-derivative?

$F(x) = \sin(x)$.

$F(x) = \sin(x) + 2$

$F(x) = \sin(x) + C$

where C is a constant.

Example Find an anti-derivative of $2x\cos(x^2)$.

$F(x) = \sin(x^2) + C$. Guess and check.

Function	Derivative
x^r	rx^{r-1}
e^x	e^x
$\ln(x)$	$1/x$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\arctan(x)$	$\dfrac{1}{1+x^2}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\mathrm{arcsec}(x)$	$\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$f(x)+g(x)$	$f'(x)+g'(x)$
$cf(x)$	$cf'(x)$

Assume F is an anti-derivative of F and g is an anti-derivative of G.

Function	An anti-derivative
$x^r, r \neq 1$	$\frac{x^{r+1}}{r+1}, \quad r \neq -1$
$1/x$	$\ln(\lvert x\rvert)$
e^x	e^x
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
$\sec^2(x)$	$\tan(x)$
$\sec(x)\tan(x)$	$\sec(x)$
$\dfrac{1}{1+x^2}$	$\arctan(x)$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$
$\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\operatorname{arcsec}(x)$
$cf(x)$	$cF(x)$
$f(x)+g(x)$	$F(x)+G(x)$

Anti-derivative of x^r:

Know $\dfrac{d}{dx} x^r = r x^{r-1}$!

$$\frac{d}{dx} x^{r+1} = (r+1) x^r$$

$$\frac{d}{dx}\left(\frac{1}{(r+1)} x^{r+1}\right) = x^r.$$

$\dfrac{x^{r+1}}{r+1}$ is an anti-derivative

of x^r, if $r+1 \neq 0$ or
$r \neq -1$.

$$\frac{d}{dx} \ln(x) = 1/x, \quad x > 0$$

If $x < 0$, $|x| = -x$.

$$\frac{d}{dx} \ln(-x) = \frac{-1}{-x} = \frac{1}{x}, \quad x < 0$$

Combining $\quad \frac{d}{dx} \ln|x| = \frac{1}{x}.$

<u>Example</u>

$$\frac{d}{dx} \ln(cx) = \frac{c}{cx} = \frac{1}{x}.$$

$$\frac{d}{dx} \ln(cx) = \frac{d}{dx}\left(\ln(x) + \ln(c)\right).$$

Notation / terminology.

Anti-derivatives are also called indefinite integrals. We write

$$\int f(x)\,dx = F(x) + C$$

if F is an anti-derivative for f.

~~How many an~~

All anti-derivatives.

Recall.

Theorem 1. If $f'(x) = 0$ for all x in an interval, then f is constant.

Why? Pick 2 points in the interval, x, y. Use MVT.

$$\frac{f(x) - f(y)}{x - y} = f'(c) = 0$$

<u>Corollary</u> If F and G are 2 anti-derivatives of f on an interval, then

$$F(x) = G(x) + C$$

for some constant C.

Example. Find all anti-derivatives of $f(x) = \frac{1}{x^2}$.

Answer $\frac{1}{x^2} = x^{-2}$

$$\int x^{-2}\,dx = \frac{x^{-2+1}}{-2+1} + C$$

$$= -\frac{1}{x} + C.$$

Is this all?

$\int \frac{1}{x^2} \, dx.$

Domain of $\frac{1}{x^2}$ is not an interval, it is $(-\infty, 0) \cup (0, \infty)$.

Most general anti-derivative is

$$F(x) = \begin{cases} \frac{1}{x} + c_1, & x > 0 \\ \frac{1}{x} + c_2, & x < 0. \end{cases}$$

Recovering a function.

~~Find an anti-derivative~~

~~of $f(x) = x^2$ with~~

~~$f(2) = 1$.~~

Find a function f with

$f'(x) = x^2$ and $f(2) = 1$.

<u>Solution</u>. General antiderivative

of x^2 is

$$f(x) = \frac{x^3}{3} + C$$

Solve for C using $f(2) = 1$.

$$\cancel{4}\ \frac{8}{3} + C = 1.$$

$$C = 1 - \frac{8}{3} = -\frac{5}{3}.$$

$$f(x) = \frac{1}{3}\cancel{(x)}^3 \cancel{+C}$$

$$= \frac{1}{3}x^3 - \frac{5}{3}.$$

Find a function $F(x)$

with $F''(x) = \cos(x)$

$F(0) = 1$

$F'(0) = 2.$

$$F'(x) = \int \cos(x)\,dx$$

$$= \sin(x) + C.$$

$$F(x) = \int \sin(x) \cancel{dx} + C\,dx$$

$$= -\cos(x) + Cx + D$$

$$F(x) = -\cos(x) + Cx + D$$

$$F(0) = 1, \quad F'(0) = 2$$

$$F(0) = -1 + 0 + D = 1.$$

$$F'(0) = \sin(0) + C = 2.$$

$$D = 2, \quad C = 2$$

$$\underline{F(x) = -\cos(x) + 2x + 2.}$$

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$F''(x) = -\sin(x)$, $F(0) = F'(0) = 0$

Solution:

$$F'(x) = \int F''(x)\,dx$$

$$= \int -\sin(x)\,dx$$

$$= \cos(x) + C$$

$$F(x) = \int \cos(x) + C\,dx$$

$$= \sin(x) + Cx + D$$

$F(0) = D = 0.$

$F'(0) = C + 1 = 0$, $C = -1.$

$$F(x) = -x + \sin(x).$$