Exam 3

Curve 4.0 11/20/15

Mean 71.?

FTC pt 1

Theorem. Let f be a continuous function on an interval [a, b], and let F be an anti-derivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

FTC = Fundamental Theorem of Calculus.

- Gives the connection between integral + the derivative.
- Useful for computing defn integrals.

Example Compute

$$\int_1^3 x^3\,dx.$$

An anti-derivative of x^3

$\frac{x^4}{4} + 101$. Use FTC pt 1

$$\int_1^3 x^3\,dx = \frac{3^4}{4} + \cancel{101} - \left(\frac{1}{4} + \cancel{101}\right) = \frac{80}{4}$$

$$= 20.$$

Why is this true?

Start with $F(b) - F(a)$.

Choose a partition

$a = x_0 < x_1 < x_2 \cdots < x_n = b$.

Apply MVT on $[x_{i-1}, x_i]$

$$F(x_i) - F(x_{i-1}) = F'(t_i)(x_i - x_{i-1}).$$
$$= f(t_i)(x_i - x_{i-1}).$$

$$F(b) - F(a) = F(x_n) - F(x_{n-1}) + F(x_{n-1}) -$$
$$\cdots + F(x_1) - F(x_0)$$

MVT

$$= \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

Take a limit as norm of the partition goes to 0.

Obtain

$$F(b) - F(a) = \int_a^b f(x)\,dx \; !$$

Notation. Evaluating F on [a, b] means

$$F(x)\Big|_{x=a}^{b} = F(b) - F(a).$$

Example.

$$\sqrt{x}\,\Big|_9^4 = \sqrt{4} - \sqrt{9}$$

$$= 2 - 3 = \underline{-1}$$

Clicker #2

$$\left. ab \right|_{a=1}^{b} = b^2 - a$$

$$\left. ab \right|_{a=1}^{b} = b \cdot b - b$$

$$= b^2 - b.$$

Try a few more

$$\int_0^{\pi} \sin(x)\,dx$$

$$= -\cos(x)\Big|_{x=0}^{\pi}$$

$$= -\cos(\pi) - {}^{-}\cos(0)$$

$$= -(-1) + 1 = \underline{2}$$

$$\int_1^4 \frac{2x^2+1}{\sqrt{x}}\,dx$$

$$= \int_1^4 x^{-1/2}(2x^2+1)\,dx$$

$$= \int_1^4 2x^{3/2} + x^{-1/2}\,dx$$

$$= 2\frac{2}{5}x^{5/2} + 2x^{1/2}\Big|_{x=1}^{4}$$

$$= \frac{4}{5}\left(4^{5/2}\right) + 2\cdot 4^{1/2} - \frac{4}{5} - 2$$

$$= \cancel{\frac{1024}{5}}$$

$$= \frac{128}{5} + 4 - \left(\frac{14}{5}\right).$$

$$= \frac{148}{5} - \frac{14}{5} = 134/5$$

__Example__ Find

$$\int_0^1 \frac{1}{x^2+1}\,dx$$

$$= \arctan(x) \Big|_{x=0}^{1}$$

$$= \frac{\pi}{4} - 0 = \pi/4.$$

Clicker #3

$$\int_{\pi}^{2\pi} \sin(x)\,dx$$

$$= -\cos(x) \Big|_{x=\pi}^{2\pi}$$

$$= -\cos(2\pi) - {}^{-}\cos(\pi)$$

$$= -1 + {}^{-}1 = -2$$

Example

Find $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sin(k/n)$.

Solution. which Riemann sum? Try $= [0,1]$.

$x_k = k/n$, $\Delta x = \frac{1}{n}$.

$= \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x\, f(x_k)$

for $f(x) = \sin(x)$.

$$= \int_0^1 \sin(x)\,dx$$

$$= -\cos(x) \Big|_0^1$$

$$= -\cos(1) + 1 .$$

Clicker #4.

Express as an integral.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} e^{2k/N}.$$

Try $[a,b] = [0,1]$.

$\Delta x = 1/N$, $x_k = k/N$.

$$= \int_0^1 e^{2t}\,dt$$

— or —

$[a,b] = [0,2]$. $x_k = 2k/N$

$\Delta x = 2/N$.

So $\frac{1}{N}\sum_{k=1}^{N} e^{2k/N} = \frac{1}{2}\,\frac{2}{N}\sum_{k=1}^{N} e^{2k/N}$

$$= \frac{1}{2}\sum_{k=1}^{N} \Delta x\, e^{x_k}$$

$$\lim \frac{1}{2}\sum_{k=1}^{N} \Delta x\, e^{x_k} = \frac{1}{2}\int_0^2 e^x\, dx$$