One more bit from substitution. 12/2/15.

$$\int \frac{dx}{\sqrt{9-x^2}}.$$

$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$ so

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$$

Substitute $x = 3u$, $dx = 3du$

$$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{3du}{\sqrt{9-9u^2}}$$

$$= \int \frac{3du}{\sqrt{9}\sqrt{1-u^2}}$$

$$= \int \frac{du}{\sqrt{1-u^2}} = \arcsin(u) + C_1$$

$$\int \frac{dx}{\sqrt{9-x^2}} = \arcsin(x/3) + C.$$

$$\int \frac{dx}{x^2 - 2x + 8}$$

$$x^2 - 2x + 8 = \cancel{x^2 - 2x + 4}$$

$$x^2 - 2x + 1 - 1 + 8$$

$$= (x-1)^2 + 7.$$

Substitute $(x-1) = \sqrt{7}\, u$

$$dx = \sqrt{7}\, du$$

⋮

$$\int \frac{dx}{(x-1)^2+7} = \int \frac{\sqrt{7}\,du}{7u^2+7}$$

$$= \frac{1}{\sqrt{7}} \int \frac{du}{u^2+1}. = \int \frac{\sqrt{7}}{(\sqrt{7})^2} \frac{du}{(u^2+1)}$$

$$= \frac{1}{\sqrt{7}} \arctan(u) + C$$

$$= \frac{1}{\sqrt{7}} \arctan\left(\frac{x-1}{\sqrt{7}}\right). + C.$$

$$\frac{1}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \cancel{\frac{\sqrt{7}}{7}} \quad \frac{\sqrt{7}}{7}.$$

Exponential Growth.

Population model.

$P(t)$ = population of critters.

Birth rate 3%

Deathrate of 2%

· Net rate is 1%. per time period.

If $P(t)$ is the population at time t,

$$P(t+h) - P(t) = 0.01\,P(t)\,h.$$

$$\frac{P(t+h)-P(t)}{h} = 0.01\,P(t).$$

Take limit as $h \to 0$

$$P'(t) = 0.01\, P(t).$$

Our 1st differential equation.

One solution is

$$P(t) = e^{0.01t}.$$

Theorem. ~~The equation~~ ~~$y' = ky$~~ All solutions of $y' = ky$ are of the form $\underline{y(t) = Ae^{kt}}$. Here A is a constant.

why?

We want to show that if $y' = ky$, then $\frac{y(t)}{e^{kt}}$ is constant. Need to show $\left(\frac{y}{e^{kt}}\right)' = 0$

$$\frac{y'e^{kt} - yke^{kt}}{e^{2kt}} = \frac{kye^{kt} - yke^{kt}}{e^{2kt}}.$$

$$= 0.$$

So $\frac{y}{e^{kt}} = A.$!

Model fitting.

Suppose y obeys

$y' = ky$, $y(1) = 2$, $y(2) = 5$.

Find $y(t)$.

Know $y(t) = Ae^{kt}$.

$\cancel{y(1) = Ae^{2k}}$

$\cancel{y(2) = A}$

Know $y(t) = Ae^{kt}$.

$y(1) = 2,\ y(2) = 5$.

$y(1) = Ae^{k} = 2$ (1)

$y(2) = Ae^{2k} = 5$.

Divide 2^{nd} eqn. by 1^{st}

$$\frac{\cancel{A}e^{2k}}{\cancel{A}e^{k}} = 5/2$$

$$e^{k} = 5/2$$

$$\ln(e^{k}) = \ln(5/2)$$

$$k\,\cancel{\ln(e)} = \ln(5/2).$$

Now find A.

$k = \ln(5/2)$. By eq. (1)

$$A e^{\ln(5/2)} = 2$$

$$A = 2/e^{\ln(5/2)}$$

$$= 2/(5/2)$$

$$= 4/5.$$

$$y(t) = \frac{4}{5} e^{t \ln(5/2)}$$

$$= \frac{4}{5}\left(\frac{5}{2}\right)^t.$$

Clicker

$$P(t) = Ae^{kt}.$$

$$P(2) = 7, \quad P(5) = 11.$$

~~P(t)~~ Find k.

$$11 = e^{k5}$$

$$7 = e^{k\cdot 2}$$

$$\frac{11}{7} = e^{3k}.$$

$$\ln(11/7) = 3k$$

$$k = \frac{1}{3}\ln(11/7).$$

$k<0$. Exponential decay.

Doubling time.

If a population grows exponentially, we can find a time T so that

$$P(t+T) = 2P(t).$$

T is called a doubling time. Find T by solving

$$\cancel{A}e^{k(t+T)} = 2\cancel{A}e^{kt}.$$

$$\cancel{e^{kt}}\, e^{kT} = \cancel{e^{kt}} \cdot 2$$

$$e^{kT} = 2.$$

or $$\ln(e^{kT}) = \ln(2)$$

$$kT = \ln(2)$$

$$T = \ln(2)/k.$$