1 Maximum-minimum problems

- Tests for absolute extrema
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1.1 Tests for absolute extrema

In section 3.1, we stated the extreme value theorem which says that every continuous function on a closed interval has an absoute maximum and absolute minimum. Based on this, we gave a procedure for finding the extreme values which is in box (8) in section 3.1.

There are two more useful tests.

Theorem 1 *(The first derivative test for absolute maximum.) Suppose that f is defined and continuous on an interval containing c. If $f'(x) > 0$ if $x > c$ and $f'(x) < 0$ if $x < c$, then c is an absolute minimum.*

Exercise. Restate this test to give a test for an absolute minimum.

A closely related test is the second derivative test. This test does not appear in the text. It states that if a function is concave up, then a critical number is an absolute minimum. More precisely, we have:

Theorem 2 *(Second derivative test for absolute extreme values.) Let f be a twice differentiable function defined on an interval and suppose for some c in the interval, $f'(c) = 0$ and that $f''(x) > 0$ for all x in the interval, then f has an absolute minimum at c.*

Proof. To prove this observe that since f'' is positive, then f' is increasing. Since $f'(c) = 0$, we can conclude that $f'(x) > 0$ for $x > c$ and that $f'(x) < 0$ for $x < c$. Thus, c is a local minimum by the first derivative test. ∎

1.2 Strategy

In the examples below, we will follow the following rough guidelines.

1. Read the problem carefully, identify the quantity that we want to make as large or small as possible. This quantity is called the "objective function".
2. Draw a diagram and introduce variables for all quantities from the problem.

3. Write an expression for the objective function. This expression may involve more than one variable.

4. Write relationships among the quantities in the problem and use them to eliminate extra variables from our objective funciton.

5. Write clearly the function (of one variable) to be optimized and state the domain. (The domain for a particular problem may be smaller than the natural domain where the function is defined.)

6. Find the extreme value of the objective funciton using one of the tests above. Explain why you know you have found the maximum.

7. Answer the question. (Are you to give where the maximum occurs, or the extreme value?)

1.3 Examples

Example. Suppose the product of two positive numbers is 5. What is the smallest possible value for the product? The smallest possible value?

Solution. (Minimum value) We let a and b be the two numbers. We are told that a and b satisfy the equation $ab = 5$. Our goal is to maximize the objective function $a + b$.

We solve $ab = 5$ to express b in terms of a (Does it matter if we solve for b or a?) which gives $b = 5/a$. Substituting for b, we obtain the objective function:

$$f(a) = a + \frac{5}{a}$$

and we want to find the smallest value of f for a in $(0, \infty)$.

If we compute the derivatives we obtain:

$$f'(a) = 1 - \frac{5}{a^2} \qquad f''(a) = \frac{10}{a^3}.$$

Thus $a = \sqrt{5}$ satisfies $f'(\sqrt{5}) = 0$ and $f''(x) > 0$ for all $x > 0$. We may use the second derivative test for absolute minimum to conclude that $a = \sqrt{5}$ is an absolute minimum on $(0, \infty)$.

The question asks for the minimum value of the sum. This is $f(\sqrt{5}) = 2\sqrt{5}$.

(Maximum value) A sketch the graph of f will indicate that f does not have a maximum value. In fact,

$$\lim_{a \to 0^+} f(a) = \lim_{a \to \infty} f(a) = +\infty.$$

Thus, the sum can be arbitrarily large and does not attain a maximum value. ∎

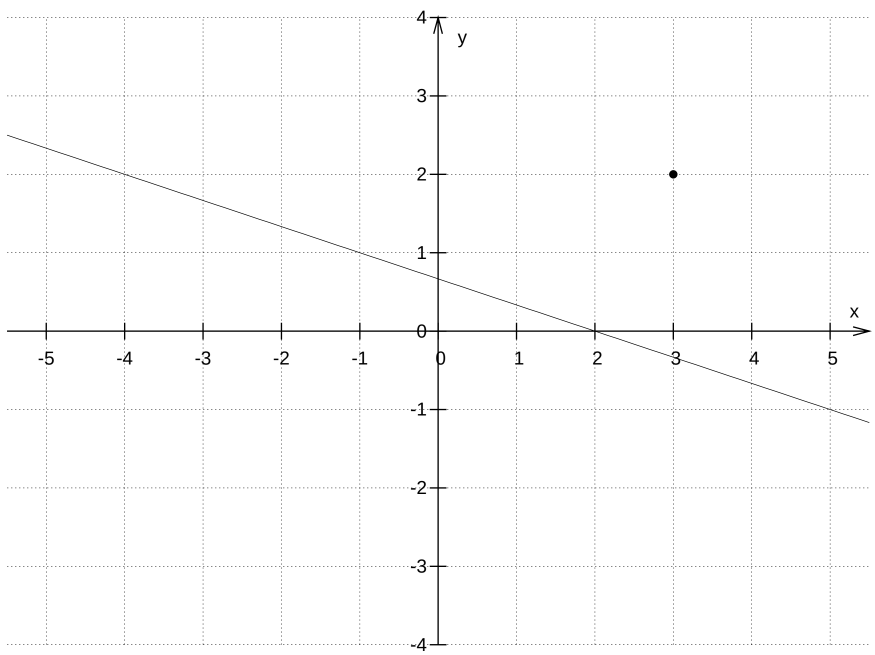
Example. Find the point on the line $x + 3y = 2$ wich is closest to $(2, 3)$.

Solution. We let (x, y) be an arbitrary point in the plane and then use the distance formula to write down the distance between (x, y) and $(2, 3)$.

$$d = \sqrt{(x-2)^2 + (y-3)^2}.$$

If the point (x, y) is to lie on the line, then we have the relation between the coordinates x and y: $x + 3y = 2$. We can solve this equation to give $x = 2 - 3y$ and substituting for x in d, gives

$$d = \sqrt{((2-3y)-2)^2 + (y-3)^2}.$$



A trick that will simplify the calculations below is to realize that the minimum value of d and the minimum value of d^2 occur at the same point. Thus, our goal is find the minimum value of

$$\begin{aligned} f(y) = d^2 &= ((2-3y)-2)^2 + (y-3)^2 \\ &= 9y^2 + y^2 - 6y + 9 \\ &= 10y^2 - 6y + 9 \end{aligned}$$

for y in $(-\infty, \infty)$.

Computing the derivatives, $f'(y) = 20y - 6$ and $f''(y) = 20$. Thus, we can use the second derivative test for absolute extreme values to conclude that the minimum occurs when $y = 3/10$. Since $x = 2 - 3y$, the nearest point will be $(x, y) = (11/10, 3/10)$.

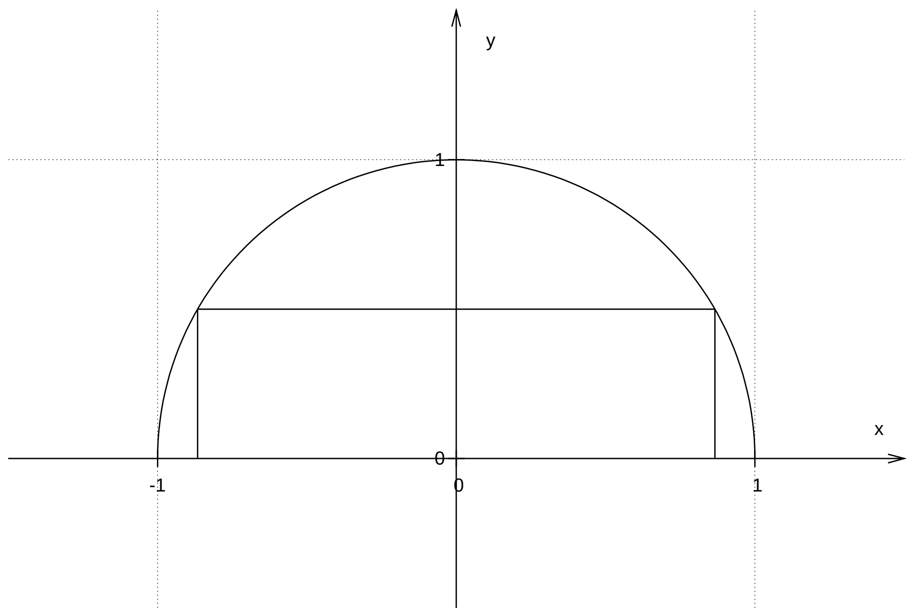
∎

Remark. It is really overkill to use calculus to find the minimum value of a parabola. A method that uses an appropriate level of force is to complete the square:

$$\begin{aligned} 10y^2 - 6y + 9 &= 10(y^2 - \frac{6}{10}y + \frac{9}{25}) - \frac{9 \cdot 10}{25} \\ &= 10((y - \frac{3}{5})^2 + \frac{108}{25}. \end{aligned}$$

Since a square of a real number is always positive, we conclude that the minimum value occurs when $y = 3/5$. ∎

Example. One side of a rectangle rests on the x-axis and two vertices touch the circle $x^2 + y^2 = 1$. Find the largest possible area.



Solution. We suppose that the vertex in the first quadrant is (x, y), then the area of the rectangle is

$$A = 2xy.$$

Because the point (x, y) lies on the circle, we have the relation $x^2 + y^2 = 1$. We may solve this equation to eliminate one of the variables, let us pick y. Then, $y = \sqrt{1 - x^2}$ so that

$$A(x) = 2x\sqrt{1 - x^2}$$

and we need to find the maximum value of the objective funciton $A(x)$ on the interval $[0, 1]$. The domain is restricted to $[0, 1]$ since the point (x, y) was chosen in the first quadrant.

Thus, we use the procedure for finding the maximum value of a continuous function on a closed interval. We test the endpoints, 0, 1, and the critical number $1/\sqrt{2}$. The maximum occurs at $x = 1/\sqrt{2}$. The largest area is 1. ∎

Second solution. When working with circles, one can also use trigonometric functions. If we let the vertex in the first quadrant be $(\cos\theta, \sin\theta)$, then our goal is find the maximum value of

$$A(\theta) = 2\cos\theta\sin\theta.$$

This maximum occurs when $\theta = \pi/4$. ∎