1 Lecture 03: Inverse functions

- The exponential and logarithm functions
- The number e and the natural logarithm
- Solving equations involving exponential functions

1.1 Exponential and logarithms

An important example of a function and its inverse are the exponential and logarithm functions. If $a > 0$ and $a \neq 1$, we define the *exponential function with base* a by $f(x) = a^x$. We are familiar with the powers $a^n = a \cdots a$ as repeated multiplication if $n = 1, 2, \ldots$ and $a^{-n} = 1/a^n$ if $n = 1, 2, \ldots$. (We can define the function 1^x, too, but it is not very interesting.) Since $(a^{1/n})^n = a$, we want $a^{1/n}$ to be the nth root $\sqrt[n]{a}$ and finally we can put $a^{m/n} = (a^{1/n})^n$. However the definition of a^x for x an irrational number is more subtle. By the end of this course, we will be able to say more about this. Altogether, we have the exponential function $f(x) = a^x$ is defined for x in the domain $(-\infty, \infty)$ and has range $(0, \infty)$.

The exponential function satisfies the properties

$$\begin{aligned} a^0 &= 1 && (1)\\ a^1 &= a && (2)\\ a^x a^y &= a^{x+y} && (3)\\ a^{-x} &= 1/a^x && (4)\\ (a^x)^y &= a^{xy} && (5)\\ & && (6) \end{aligned}$$

The inverse function to this exponential function is called the logarithm with base a, denoted by $\log_a$.

Example. Find the values of $\log_{10}(100)$ and $\log_2(\sqrt{2})$.

Solution. Since the function 10^x takes 2 to 100, the inverse function takes 100 to 2, $\log_{10}(100) = 2$.

Since $\sqrt{2} = 2^{1/2}$, we have that $\log_2(\sqrt{2}) = 1/2$. ∎

Each of the properties of the exponential function can be recast as a property of the logarithm function.

$$\begin{aligned} \log_a(1) &= 0 && && (7)\\ \log_a(a) &= 1 && && (8)\\ \log_a(x) + \log_a(y) &= \log_a(xy), && x > 0, y > 0 && (9)\\ \log_a(1/x) &= -\log_a(x), && x > 0 && (10)\\ \log_a(x^r) &= r\log_a(x), && x > 0, r \in (-\infty, \infty). && (11) \end{aligned}$$

To see why (9) is true, we can write $x = a^{\log_a(x)}$, $y = a^{\log_a(y)}$, and $xy = a^{\log_a(xy)}$. Then using property (3) of the exponential function, we have

$$a^{\log_a(xy)} = xy = a^{\log_a(x)} a^{\log_a(y)} = a^{\log_a(x)+\log_a(y)}.$$

Since the exponential function is one-to-one, we have $\log_a(xy) = \log_a(x) + \log_a(y)$.

1.2 The number e

As we saw in the previous section, there is a different logarithm $\log_a$ for each $a > 0$ (except $a = 1$). Which one is best? For the purposes of calculus, we use a special number $e \approx 2.781828...$ The logarithm and exponential function for this base are written as

$$e^x \text{ or } \exp(x) \qquad \text{and} \qquad \ln(x)$$

and $\ln(x)$ is called the natural logarithm. We will see why this logarithm is natural. At the moment, it is a puzzle why we use the base e instead of a more familiar number such as 2 or 10. However, we will see that doing calculus with the function e^x is particularly easy and this explains why we prefer this base.

It is useful to observe that any exponential function can be expressed in terms of the function e^x.

Example. Write 4^x in the form e^{rx}.

Solution. We want to have $4^x = e^{rx}$. If we take the natural log of both sides, we have $x\ln(4) = rx$ or $r = \ln(4)$. Thus we have $4^x = e^{x\ln(4)}$. ∎

Example. If x and y are positive numbers and $\ln(xy^2) = 2$ and $\ln(x/y) = 0$, find x and y.

Solution. If we write $x = e^a$ and $y = e^b$, (actually $a = \ln(x)$ and $b = \ln(y)$ then we have

$$2 = \ln(e^a e^{2b}) = (a + 2b)\ln(e) = (a + 2b)$$

and

$$0 = \ln(e^a/e^b) = \ln(e^{a-b}) = a - b.$$

Solving the system of equations

$$a + 2b = 2, \qquad a - b = 0$$

gives $a = 2/3$ and $b = 2/3$ or $x = y = e^{2/3}$.

And of course we can check our answers. ∎

We know the square root function is very useful for solving quadratic equations. We see how the logarithm can be used to solve equations involving exponentiation.

Example. A function f is of the form $f(t) = Ae^{kt}$. Will k be positive or negative?

Find A and k so that $f(2) = 11$ and $f(5) = 4$.

Solution. Since $f(5) < f(2)$, we expect that f is decreasing and thus $k < 0$.

To solve the value of A and k, observe that the given conditions on f gives us the equations

$$11 = Ae^{2k}, \qquad 4 = Ae^{5k}.$$

We can solve each equation for A and obtain

$$A = 11e^{-2k} \qquad A = 4e^{-5k}.$$

Equating the values of A, we have $11e^{-2k} = 4e^{-5k}$ or that $e^{3k} = 4/11$. To solve this equation, we apply the natural log to both sides and obtain

$$\ln(4/11) = \ln(e^{3k}) = 3k\ln(e) = 3k.$$

or $k = \frac{1}{3}\ln(4/11)$. Since the natural log of a number less than 1 is negative, we have $k < 0$ as expected. Finally, we have $A = 11e^{-2\frac{1}{3}\ln(4/11)} = 11(4/11)^{-2/3}$. Thus we have

$$A = 11^{5/3}4^{2/3} \approx 21.591 \qquad k = \frac{1}{3}\ln(4/11) \approx -0.3372.$$

We may check our answer by computing $21.591 \cdot e^{-0.3372 \cdot 2} \approx 11$.

∎

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