1 Lecture 37: The fundamental theorems of calculus.

- The fundamental theorems of calculus.
- Evaluating definite integrals.
- The indefinite integral-a new name for anti-derivative.

Today we provide the connection between the two main ideas of the course. The integral and the derivative.

Theorem 1 *(FTC I) Suppose f is a continuous function on $[a, b]$. If F is an anti-derivative of f, then*

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Example. Compute

$$\int_0^3 x^3\,dx.$$

Solution. We choose our favorite anti-derivative of x^3, say $F(x) = x^4/4 + 101$. We have

$$\int_0^3 x^3\,dx = \frac{3^4}{4} + 101 - (\frac{0}{4} + 101) = 81/4.$$

∎

We give an idea of the proof.

Proof. We let F be an anti-derivative of f and let $P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$. We will express the change of F, $F(b) - F(a)$, as a Riemann sum for this partition. Letting the size of the largest interval in the partition tend to zero, we obtain the integral is equal to the change in F.

We begin by writing

$$F(b) - F(a) = F(x_n) - F(x_{n-1}) + F(x_{n-1}) - \ldots + F(x_i) - F(x_{i-1}) + \ldots + F(x_1) - F(x_0).$$

We recall that F is an anti-derivative of f and apply the mean value theorem on each interval $[x_{i-1}, x_i]$ and find a value c_i so that $F(x_i) - F(x_{i-1}) = f(c_i)(x_i - x_{i-1})$. Thus, we have

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}).$$

Since the right-hand side is a Riemann sum for the integral, we may let the width of the largest subinterval tend to zero and obtain

$$F(b) - F(a) = \int_a^b f(s)\,ds.$$

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1.1 Indefinite integrals.

We use the symbol

$$\int f(x)\,dx$$

to denote the indefinite integral or anti-derivative of f. This should include a constant C to indicate that the choice of indefinite integral involves fixing an arbitrary constant.

The indefinite integral is a function. The definite integral is a number. According FTC I, we can find the (numerical) value of a definite integral by evaluating the indefinite integral at the endpoints of the integral. Since this procedure happens so often, we have a special notation for this operation.

$$F(x)|_{x=a}^{b} = F(b) - F(a).$$

Example. Find

$$xa|_{x=a}^{b} \qquad \text{and} \qquad xa|_{a=x}^{y}$$

Solution.

$$ba - a^2 \qquad xy - x^2$$

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Example. Verify

$$\int x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2) + C.$$

Solution. According to the definition of anti-derivative, we need to see if

$$\frac{d}{dx}\frac{1}{2}\sin(x^2) = x\cos(x^2).$$

This holds, by the chain rule. ∎

1.2 Computing integrals.

The main use of FTC I is to simplify the evaluation of integrals.

We give a few examples.

Example. a) Compute

$$\int_0^{\pi} \sin(x)\,dx.$$

b) Compute

$$\int_1^4 \frac{2x^2+1}{\sqrt{x}}\,dx.$$

c) Find

$$\int_0^1 \frac{1}{1+x^2}\,dx.$$

Solution. a) Since $\frac{d}{dx}(-\cos(x)) = \sin(x)$, we have $-\cos(x)$ is an anti-derivative of $\sin(x)$. Using the second part of the fundamental theorem of calculus gives,

$$\int_0^\pi \sin(x)\,dx = -\cos(x)|_{x=0}^{\pi} = 2.$$

b) We first find an anti-derivative. As the indefinite integral is linear, we write

$$\int \frac{2x^2+1}{\sqrt{x}}\,dx = \int 2x^{3/2} + x^{-1/2}\,dx \quad = \quad 2\int x^{3/2}\,dx + \int x^{-1/2}\,dx = \frac{4}{5}x^{5/2} + 2x^{1/2} + C.$$

With this anti-derivative, we may then use FTC I to find

$$\begin{aligned} \int_1^4 \frac{2x^2+1}{\sqrt{x}}\,dx &= \left.\frac{4}{5}x^{5/2} + 2x^{1/2}\right|_{x=1}^{4} \\ &= \frac{4}{5}4^{5/2} + 24^{1/2} - (\frac{4}{5} + 2) \\ &= 128/5 + 20/5 - (4/5 + 10/5) \\ &= 134/5. \end{aligned}$$

c) We recall that $\arctan(x)$ is an anti-derivative of $1/(1+x^2)$ and thus

$$\int_0^1 \frac{1}{1+x^2}\,dx = \arctan(x)|_{x=0}^{1} = \arctan(1) = \pi/4.$$

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Example. Find

$$\int_0^{\sqrt{\pi}} 2x\cos(x^2)\,dx.$$

Solution. We recognize that $\sin(x^2)$ is an anti-derivative of $2x\cos(x^2)$,

$$\int 2x\cos(x^2)\,dx = \sin(x^2) + C.$$

Thus,

$$\int_0^{\sqrt{\pi}} 2x\cos(x^2)\,dx = \sin(x^2)\Big|_{x=0}^{\sqrt{\pi}} = 0 - 0.$$

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Note that we needed a certain amount of luck to find this anti-derivative. One of the goals in the future is to learn techniques for finding anti-derivatives in general.

Here, is a more involved example that illustrates the progress we have made.

Example. Find

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \sin(k/n).$$

Solution. We recognize that

$$\frac{1}{n}\sum_{k=1}^{n}\sin(k/n)$$

is a Riemann sum for an integral. The points x_k, $k = 0, \ldots, n$ divide the interval $[0, 1]$ into n equal sub-intervals of length $1/n$. Thus, we may write the limit as an integral

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sin(k/n) = \int_0^1 \sin(x)\,dx.$$

To evaluate the resulting integral, we use FTC I. An anti-derivative of $\sin(x)$ is $-\cos(x)$, thus

$$\int_0^1 \sin(x)\,dx = -\cos(x)\big|_{x=0}^{1} = 1 - \cos(1).$$

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