1 Lecture 38: The fundamental theorems of calculus.

- The second part of the fundamental theorem of calculus.
- Differentiating integrals.
- Recovering a function from its rate of change.

Warmup question: Does the function e^{-x^2} have an anti-derivative? Can you find it?

1.1 Differentiating integrals.

Theorem 1 *(FTC II) Assume f is continuous on an open interval I and a is in I. Then the area function*

$$A(x) = \int_a^x f(t)\,dt$$

is an anti-derivative of f and thus $A' = f$.

The most of important consequence of FTC II is that any continuous function has an anti-derivative. We will also work exercises where we apply FTC II to differentiate functions defined by integrals.

Proof. Write

$$\frac{A(x+h) - A(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

We will show

$$\lim_{h\to 0^+} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x).$$

The reader should write out a similar argument for the limit from the left.

If f is continuous, then f has maximum and minimum values M_h and m_h on the interval $[x, x+h]$. Using the order property of the integral,

$$m_h \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le M_h.$$

As h tends to 0, we have $\lim_{h\to 0^+} M_h = \lim_{h\to 0^+} m_h = f(x)$ since f is continuous. It follows that

$$\lim_{h\to 0^+} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x).$$

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Example. Find

a) $\frac{d}{dx}\int_0^x \sin(t^2)\,dt$

b) $L'(x)$if $L(x) = \int_1^{x^2} \frac{1}{t}\,dt$. Is the function $L(x) = \int_1^x \frac{1}{t}\,dt$ increasing or decreasing? Is the graph of L concave up or concave down?

c) $\frac{d}{dx}\int_{x^2}^x \sin(t^2)\,dt$

Solution. Part a) is a straightforward application of the second part of the fundamental theorem. The function $\sin(x^2)$ is continuous everywhere and thus we have

$$\frac{d}{dx}\int_0^x \sin(t^2)\,dt = \sin(x^2).$$

To work part b), we let $F(u) = \int 1^u \frac{1}{t}\,dt$. By FTC II, we have $F'(u) = 1/u$. Now we may write $L(x) = F(x^2)$ and then $L'(x) = F'(x^2)2x$ by the chain rule. Thus

$$\frac{d}{dx}\int_1^{x^2} = \frac{2x}{x^2} = \frac{2}{x}, \qquad x > 0.$$

A second approach is to use FTC I to see that $\int_1^x \frac{1}{t}\,dt = \ln(x^2) - \ln(1) = 2\ln(x)$ and then apply the differentiation rules to compute the derivative. Note that we could not use this approach in the first example since we do not know an anti-derivative for $\sin(x^2)$.

We know that L is increasing since $F'(x) > 0$. Taking another derivative, we find that

$$\frac{d^2}{dx^2}L(x) = -2/x^2.$$

Thus this function is concave down for $x > 0$.

Finally, part c) requires us to use the properties of the integral to put it in a form where we can use FTC II. We can write

$$\int_{x^2}^x \sin(t^2)\,dt = \int_{x^2}^0 \sin(t^2)\,dt + \int_0^x \sin(t^2)\,dt = -\int_0^{x^2} \sin(t^2)\,dt + \int_0^x \sin(t^2)\,dt.$$

Now applying FTC II and using the chain rule for the first integral gives

$$\frac{d}{dx}(-\int_0^{x^2} \sin(t^2)\,dt + \int_0^x \sin(t^2)\,dt) = -2x\sin(x^4) + \sin(x^2).$$

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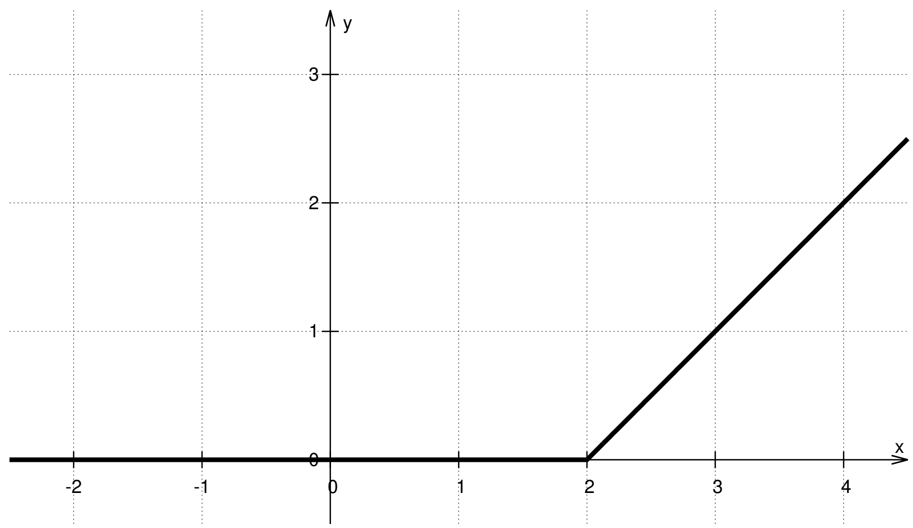

Our second example shows that it is necessary to assume that f is continuous in FTC II.

Example. Let f be the function given by

$$f(x) = \begin{cases} 0, & x < 2 \\ 1, & x \geq 2 \end{cases}$$

Find $F(x) = \int_0^x f(x)\,dx$ and determine where F is differentiable.

Solution. We have that the integral is given by

$$F(x) = \begin{cases} 0, & x < 2 \\ (x-2), & x \geq 0 \end{cases}$$

It is pretty clear that F is differentiable everywhere except at 2. At 2, we can compute the left and right limits of the difference quotient and find

$$\lim_{h \to 0^-} \frac{F(2+h) - F(2)}{h} = 0 \qquad \lim_{h \to 0^+} \frac{F(2+h) - F(2)}{h} = 1.$$

Thus $F'(2)$ does not exist.

The graph of F below confirms this. ∎

1.2 The net change theorem

Since F is always an anti-derivative of F', one consequence of part I of the fundamental theorem of calculus is that if F' is continuous on the interval $[a, b]$, then

$$\int_a^b F'(t)\, dt = F(b) - F(a).$$

This is really FTC I again, but is called the net change theorem in the text. Repeating the result with a new name serves to emphasize that the integral allows to recover the net change of a function from the rate of change. Another formulation is that if we know the initial value of F at a and the rate of change over the interval $[a, b]$, then we can find $F(b)$. This idea has many applications.

Example. An object falls with constant acceleration $-g$, at $t = 1$ its height is h_1 and its velocity is v_1. Find its position at all times.

Solution. By the net change theorem,

$$v(t) - v(1) = \int_1^t g\, ds = -g \cdot t + g = -g \cdot (t - 1).$$

Thus $v(t) = -g \cdot (t - 1) + v_1$. Applying the net change theorem again we have the height at time time t, $h(t)$ is

$$\begin{aligned} h(t) - h(1) &= \int_1^t -g \cdot (s - 1) + v_1\, ds = (-\frac{1}{2} g \cdot s^2 + g \cdot s + v_1 \cdot s)|_{s=1}^t \\ &= -\frac{1}{2} g t^2 + g t + v_1 t + \frac{1}{2} g - g - v_1 \\ &= -\frac{1}{2} g \cdot (t^2 - 2t + 1) + v_1 \cdot (t - 1) \\ &= -\frac{1}{2} g \cdot (t - 1)^2 + v_1 \cdot (t - 1). \end{aligned}$$

Thus

$$h(t) = \frac{1}{2} g \cdot (t - 1)^2 + v_1 \cdot (t - 1) + h_1.$$

∎

Note this gives a different version of the equations for a falling object from Chapter 3.

Example. Suppose that a particle moves so that its velocity at time t is $v(t) = \sin(t)$.

Find the net change in position over the interval $[0, 2\pi]$. Find the total distance traveled in the interval $[0, 2\pi]$.

Solution. The key conceptual point is to observe that the particle changes direction during the interval $[0, 2\pi]$, thus we expect that the total distance travelled will be greater than the net change in position.

To do the calculations, we first compute the change in displacement by FTC I/the Net Change Theorem $p(2\pi) - p(0) = \int_0^{2\pi} v(t)\,dt$. In this problem, we have $v(t) = \sin(t)$ and thus the change in position is

$$\int_0^{2\pi} \sin(t)\,dt = -\cos(t)|_0^{2\pi} = 0.$$

To find the distance travelled, we need to compute the areas above and below the t axis and add, rather than subtract, them to get the total distance travelled. Since the velocity $v(t) = \sin(t)$ is positive on the interval $[0, \pi]$ and negative on the interval $[\pi, 2\pi]$, we have that the total distance travelled is

$$\int_0^{\pi} \sin(t)\,dt - \int_{\pi}^{2\pi} \sin(t)\,dt = -\cos(t)|_{t=0}^{\pi} + \cos(t)|_{t=\pi}^{2\pi} = 4.$$

∎

Example. To give a less familiar example, suppose we have a rope whose thickness varies along its length. Fix one end of the rope to measure from and let $m(x)$ denote the mass in kilograms of the rope from 0 to x meters along the rope. If we take the derivative, $\frac{dm}{dx} = \lim_{h\to 0}(m(x+h) - m(x))/h$, then this represents mass per unit length (or linear density) of the rope near x and the units are kilograms/meter. If we integrate this linear density and observe that $m(0) = 0$, then we can find $m(x)$ which represents the mass of the first x meters of the rope,

$$m(x) = \int_0^x \frac{dm}{dx}\,dx.$$

This is another example of the net change theorem.

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