1.2 The principle of mathematical induction

We let $\mathbf{N} = \{1, 2, 3, 4, \dots\}$ denote the natural numbers. Let S be a set of natural numbers with the properties that

- $1 \in S$
- If $N \in S$, then $N + 1 \in S$.

The *principle of mathematical induction* states that if a set S is a set of the natural numbers and S has these properties, then all natural numbers lie in S.

We may give a definition of the expression

$$\sum_{k=1}^{n} f(k)$$

by the statements

$$\sum_{k=1}^{1} f(k) = f(1) \qquad \sum_{k=1}^{n} f(k) = f(n) + \sum_{k=1}^{n-1} f(k).$$

The principle of mathematical induction shows that this pair of statements serve to define the operation

$$\sum_{k=1}^{n} f(k)$$

for all natural numbers n.

1. Give similar definitions of the expressions a^n and $n!$ for $n = 0, 1, 2, \dots$.

2. If we make one cut through a pizza, we obtain two pieces of pizza. If we make two cuts, we will obtain three or four pieces depending on how we make the cut. What is the maximum number of pieces we can obtain if we make n cuts?

3. If we draw two lines in the plane, they may interect zero or one times. If we draw three lines, we can obtain zero, one, two or three intersections. Give examples of each possibility.

4. If we draw n distinct lines in the plane, what is the largest possible number of intersections?

5. If we draw n distinct circles in the plane, what is the largest possible number of intersections?

6. Prove that if n is a natural number and $n^2 + n$ is odd, then $(n+1)^2 + (n+1)$ is odd.

7. Find all natural numbers n for which $n^2 + n$ is odd.

8. Use mathematical induction to establish that

$$\sum_{k=1}^{n}(2k-1) = n^2, \qquad n = 1, 2, \ldots$$

9. Use mathematical induction to establish that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

10. Use mathematical induction to establish that if r is not 0 or 1 and $n = 0, 1, 2, \ldots,$ we have

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}.$$

1.3 The towers of Hanoi

In the Towers of Hanoi puzzle, we have two empty poles and a third pole with a stack of n disks which are arranged in order of decreasing size. In each move, we move one disk from one pole to another with the restriction that we may never place a larger disk on top of a smaller disk.

Is it possible to move the disks from one pole to another? Our goal is to see that this is always possible and to find the smallest number of moves that is needed to move n disks?

1. Let t_n be the minimum number of moves needed to move n disks from one pole to another. Find t_n for $n = 1, 2, 3, 4$ and 5.

2. Can you guess a formula for t_n? How can we be sure this formula is right?

3. Can you express t_{n+1} in terms of t_n? Give a few sentences to explain why your relation holds.

4. Use your answer to question 3 to compute t_n for n between 1 and 10.

5. The next few problems discuss linear recurrence relations. We begin by trying a few simple examples.

   Find a sequence s_n which satisfies

   $$s_{n+1} = 3s_n, \qquad s_0 = 1.$$

6. Find a sequence s_n which satisfies

   $$s_{n+1} = 4s_n, \qquad s_0 = 7.$$

7. Find a sequence s_n which satisfies

   $$s_{n+1} = 2s_n - 1, \qquad s_0 = 5.$$

8. Fix numbers a, b, and c and consider the problem of finding a sequence s_n for $n = 0, 1, 2, \ldots$ so that

   $$\begin{cases} s_{n+1} = as_n + b, & n = 0, 1, 2, \ldots \\ s_0 = c. \end{cases}$$

   We say that s_n satisfies a linear recurrence relation with given initial value.

   Can you find a formula for s_n?

9. Use the method of question 8 and your answer to question 3 to find a formula for t_n, the number of moves needed in the Towers of Hanoi.

10. Suppose we play the Towers of Hanoi game with 4 poles instead of 3. Will it take more moves or fewer moves than with three poles?

11. Let $T(n, k)$ be the number of moves needed to move n disks on k poles. Find $T(n, 4)$ for small values of n.

12. If we let $T(n, k)$ denote the minimum number of moves needed to move n disks with k poles, what can we say about $T(n, k)$?

    In an earlier problem, we found a formula for $T(n, 3)$.

    Explain why $T(n, 4) \leq 2T(k, 4) + T(n - k, 3)$ for $k = 1, \ldots, n - 1$.

13. Explain why we must have $T(n, k) \geq 2n - 1$ for any $k \geq 3$.

14. Can you find other lower bounds for $T(n, k)$?

    Can you find a formula for $T(n, 4)$ in terms of n?