1.4 The binomial theorem

1. Expand $(a+b)^2$, $(a+b)^3$ and $(a+b)^4$.

2. Fix a natural number n and suppose that we have constants $A_{n,k}$, $k=0,\dots,n$ so that
$$(a+b)^n = \sum_{k=0}^{n} A_{n,k} a^{n-k} b^k.$$
Write $(a+b)^{n+1} = (a+b)^n(a+b)$ and find an expansion for $(a+b)^{n+1}$ using the coefficients $A_{n,k}$.

3. What is the product of no numbers? Give two examples to support your answer.
What should the value of $A_{0,0}$ be?

4. Find the values of $A_{n,0}$ and $A_{n,n}$ for $n=1,2,\dots$.

5. Now let n vary and find the value of $A_{n,1}$ for $n=1,2,\dots$?

6. Can you find the value of $A_{4,2}$, $A_{5,2}$ and $A_{6,2}$?

7. Can you find the value of $A_{n,2}$ for $n=1,2,\dots$?

8. Compute
$$\frac{10!}{7!\cdot 6!}, \qquad \frac{1000!}{998!}.$$

9. We define the binomial coefficient
$$\binom{n}{k} = \frac{n!}{(n-k)!\cdot k!}, \qquad n=0,1,2,\dots \text{ and } k=0,\dots,n.$$
Show that we have
$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$
for $n=0,1,2,\dots$ and $k=0,\dots,n$.

10. Can you find a formula for the numbers $A_{n,k}$ for $n=0,1,2,\dots$ and $k=0,\dots,n$.
Prove that your formula is correct.

11. Find the value of the sum
$$\sum_{k=0}^{4}\binom{4}{k}3^k.$$

12. Find the value of the sum
$$\sum_{k=0}^{5}\binom{5}{k}2^k.$$

13. For $n = 0, 1, 2, \ldots$, find a simple expression for

$$\sum_{k=0}^{n} 2^k \binom{n}{k}.$$

14. For $n = 0, 1, 2, \ldots$, find a simple expression for

$$\sum_{k=0}^{n} \binom{n}{k}.$$

15. For $n = 0, 2, 4, \ldots$, find a simple expression for

$$\sum_{k=0}^{n/2} \binom{n}{2k}.$$

For $n = 0, 2, 4, \ldots$, find a simple expression for

$$\sum_{k=1}^{n/2} \binom{n}{2k-1}.$$

16. Can we find a version of the previous problem when n is odd?

17. What is the result if we fix n and sum every fourth binomial coefficient?

1.5 Some fun.

1. Find out what happens when you refer to the FOIL method in the presence of the instructor. Explain your answer.

2. Find a mnemomic similar to FOIL to help expand the product
$$(a+b+c)(d+e+f).$$

3. *Theorem:* All horses are the same color.

   Proof by induction: We let
$$S = \{n : \text{Every set with } n \text{ horses consists of horses of the same color.}\}.$$

   Clearly 1 is in S. Now suppose N is in S and consider a set A with $N+1$ horses. If we choose a subset A' of A and A' contains N horses, then all of the horses in A' are the same color. Since this is true for every subset of A which contains N elements, it follows that A also consists of horses of the same color. Thus $N+1$ is in S, when N is in S. Since 1 is in S and if N is in S, then $N+1$ is in S, the principle of mathematical induction implies that S is the set of all natural numbers.

   Since every finite set of horses contains only horses of the same color, it follow that all horses are the same color.

   Is there a flaw in the above proof? If so, where?