4 Geometry

4.1 Tiling the plane

1. Let P be a convex n-gon or polygon with n sides. Describe a method for subdividing P into triangles.

2. Using the decomposition in the previous problem, find the sum of the interior angles in a convex n-polygon. Express your answer in radians.

3. Find the measure of one interior angle in a regular pentagon, a regular hexagon and a regular heptagon.

4. Suppose that we have k regular n-gons in the plane which share a common vertex. What are the possible values of k and n?

   Which regular n-gons can tile the plane?

5. Find the sum of the interior angles for two regular hexagons and a regular pentagon which share a common vertex.

4.2 The soccer ball

1. Many soccer balls are covered with a pattern of regular hexagons and regular pentagons. You will find that a pentagon and two hexagons share each vertex. What is the sum of the interior angles of two regular hexagons and a regular pentagon in the plane? Do you notice something interesting.

2. Consider the sphere $\mathbf{S}^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. A great circle is the intersection of this sphere with a plane that passes through the origin. If we choose two points on a sphere, the shortest path on the sphere that joins these points will be a great circle.

   Every pair of points can be joined by two paths which lie on a great circle. Can you find a pair of points that are joined by more than two paths from great circles?

3. A spherical triangle T is a set of three points on the sphere and segments of great circles that join them. Can you find a triangle where the interior angles sum to more than π? (Note that we do not have a good definition of the measure of an angle on the sphere. You will have to guess. Perhaps we will return to this point.)

4. What angle sums for triangles are possible on the sphere? Find examples to illustrate your answer.

5. It is known that on a sphere of radius 1, the angle sum of a triangle T with angles a, b, and c satisfies:

   $$a + b + c = \pi + \gamma \text{area}(T).$$

   By considering some examples, find the constant of proportionality γ. (Note this is not a proof, but we need to save something for the next course.)

6. Find a relation between the angle sum for a regular hexagon on a sphere and its area. Do the same for a regular pentagon.

7. If we cover a sphere of radius 1 with the 20 hexagons and 12 pentagons as we find on a soccer ball, determine the interior angles for the hexagon and the pentagon. Hint: At a vertex on the soccer ball, the three angles have to sum to 2π. This gives us one equation. Find another equation using the previous problem.

8. Some soccer balls have a pattern consisting of pentagons, triangles and squares. See the website. Can you determine how many of each are in this pattern?

4.3 Origami and a curve in the plane

1. Take an ordinary sheet of paper, draw a straight line ℓ across the paper and choose a point P that is not on the line. Choose a point Q on ℓ and fold the point P onto Q, making a crease in the paper. Do this for several different choices of Q along the line ℓ. Try to make ten folds. What can you say about the creases you create in this way?

2. In the coordinate plane consider the line ℓ given by $y = -1$ and the point P given by $P = (0, 1)$. Make a sketch showing points R which are equidistant from ℓ and P.

   The set of all points that satisfy a given condition is often called a *locus*. The locus of all points that are equidistant to a line ℓ and a point P is called the parabola with focus P and directrix ℓ.

3. Given a line ℓ and a point P, can you use a ruler and compass to construct points on the parabola with focus P and directrix ℓ?

4. Let ℓ and P be as in problem 2. Find an equation which describes the points that are equidistant from P and ℓ.

5. Returning to problem 1. If we fold P onto a point Q on ℓ, what can you now say about the fold line?

6. Prove the conjecture made in the previous problem.

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